Viscous Conservation Laws with Boundary Layers

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We study the asymptotic limiting behavior of the solutions to the initial boundary value problem for linearized one-dimensional compressible Navier-Stokes equations. We consider the characteristic boundary conditions, i.e. we assume that an eigenvalue of the associated inviscid Euler system vanishes uniformly on the boundary. The aim of this paper is to understand the evolution of the thermal boundary layer, to construct the asymptotic ansatz which is uniformly valid up to the boundary, and to obtain rigorously the uniform convergence to the solution of the Euler equations.

提　　要

本文考虑一维线性化的可压 Navier-Stokes 方程初边值问题解的渐近性态。我们研究的是特征边界的情形，即相应的非粘性 Euler 方程的一个特征值在边界上为零。本文的主要目的在于了解该情形下温度边界层的产生，并利用多尺度渐近分析的方法构造 Navier-Stokes 方程初边值问题的近似解，最终通过能量估计方法严格证明该近似解到相应 Euler 方程初边值问题解的一致收敛性。

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Introduction

The asymptotic equivalence between a viscous parabolic system and its associated inviscid hyperbolic equations in the limit of small dissipations is of considerable significance in many physical phenomena and their numerical computations. This is particularly so in the presence of shock discontinuities and boundaries(see [29]). The problems have been much studied in the linear and semi-linear cases, for instance in [1], [2],[14] and [26]. It is well-known that for a hyperbolic equation or a system, to impose the boundary condition should be much careful, otherwise it would leads to an ill-posed problem, for details, see [17] and the references therein. However, an initial-boundary value problem for a parabolic equation or a system in a certain domain would be well-posed as soon as the initial and boundary conditions satisfy some compatibility conditions, see [6]. Thus, it is commonly believed that solutions of the viscous parabolic equations cannot be uniformly close to those of the inviscid hyperbolic equations, unless the boundary conditions are chosen in a very special way. In many cases, such discrepancies in velocity or temperature lead to the phenomena of boundary layers, which have to be resolved rigorously in mathematical analysis, for example [4], [29], [30], [32] and [33]. There are two distinguished classes of boundary conditions, which yield substantial different behavior, depending on whether the boundary is characteristic or not. In the non-characteristic case, for scalar equations, with or without boundary, the convergence of viscous solution in mean to even weak entropy solution to the inviscid hyperbolic problem has been established by using maximum principle and entropy estimates, see [3], [15]. Yet, little information is given by this approach on the asymptotic behavior of the viscous solutions for small but non-zero viscosity. Furthermore, it seems to be extremely difficult to apply such an approach to any system which do not admit weakly maximum principle. To conquer this difficulty, Xin[32] developed a new approach based on matched asymptotic analysis and an energy method. In the paper he obtained the asymptotic limiting behavior, before the possible development of singularities, of solutions to one-dimensional quasilinear scalar viscous equation

$$\partial_t u^\varepsilon + \partial_x(f(u^\varepsilon, x, t)) + g(u^\varepsilon, x, t) = \varepsilon \partial_x^2 u^\varepsilon, \tag{0.0.1}$$

for $u^\varepsilon \in \mathbb{R}^1, (x,t) \in \mathbb{R}^1_+ \times \in \mathbb{R}^1_+, \quad \varepsilon > 0$ with small viscosity in the presence of boundaries under the assumption that the boundary is non-characteristic. He showed rigorously that if the difference between the prescribed boundary value of the flow and the value of the inviscid flow at the boundary is sufficiently small (weak boundary layers), the inviscid flow plus the leading term in the formal expansion near the boundary are uniformly approximated by the viscous solution. The condition is fulfilled for small time, i.e. the short-time stability. As a consequence, it was shown that the viscous flow converges uniformly to the inviscid one away from the boundary, and the similar results are valid for the strong boundary layers. For systems, a symmetrizable hyperbolic system with a small viscosity of order ε,

$$\partial_t u + \sum_{j=0}^{n} A_j(b,u)\partial_j u + B(b,u) - \varepsilon \Delta u = f(t,x), \quad t > 0,\ x_n > 0. \tag{0.0.2}$$

was studied in [12] by Grenier and Guès under the assumption that the boundary is noncharacteristic. In their paper, the initial and boundary conditions are chosen to be zero, they also characterized the boundary conditions which survive in the limit $\varepsilon \to 0$. Then by an elaborate energy estimate, the heart of the proof, they showed how a formal approximate solution can be constructed using matched asymptotics, and this can be carried out up to any order of ε. Their results generalized earlier works on semilinear equations [7] and one-dimensional problems [14]. The linear and nonlinear stability of large multi-dimensional viscous boundary layers arising in a parabolic regularization of a hyperbolic initial-boundary value problem with non-characteristic boundary for an $n \times n$ system of multi-dimensional quasilinear hyperbolic equations were studied in [23]. By exploiting the concept of Evans functions and using a linearization technique, they gave sufficient conditions which guaranteed the stability of boundary layers. Being regarded as the generalization of the result in [8], detailing the approximation of the Green's function of the linearized equations about the formal matched expansion, Grenier and Rousset [13] presented the results on the existence and stability of large amplitude non-characteristic boundary layers for parabolic conservation laws as viscosity goes to zero, in the case of a single spatial dimension. They showed that the condition of smallness could be replaced by an accurate spectral condition on the formal boundary layer, phrased in terms of an associated Evans function. For the similar case of the compressible Navier-Stokes system, see Xin [36]. As for the case that in presence of the shock discontinuities, somewhat like the non-characteristic boundary case, Goodman and Xin [10] obtained the following elegant result. Assume that the p-th characteristic family for

$$u_t + f(u)_x = 0, \quad u \in \mathbb{R}^n,\ x \in \mathbb{R},\ t > 0. \tag{0.0.3}$$

is genuinely nonlinear and u is a piecewise smooth solution with a single, entropic, sufficiently weak p-shock. Then u is the limit of a sequence of solutions u^ε to

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon, \quad u \in \mathbb{R}^n,\ x \in \mathbb{R},\ t > 0. \tag{0.0.4}$$

as $\varepsilon \to 0$. Away from the shock, the difference between u^ε and u, measured in sup-norm, is of order ε globally, the L^2- norm of $u^\varepsilon - u$ is of order ε^η for every $\eta < 1$. The result holds as well for solutions with finitely many non-interacting entropic shocks. The ideas in this paper can also be applied to study the solutions of initial-boundary value problem for (0.0.3) and (0.0.4) and for one-dimensional compressible Navier-Stokes equations with non-characteristic boundary.

From a physical point of view, another natural assumption is that the boundary is impermeable; then this leads to characteristic boundary conditions. The characteristic case, though more physical since for example in the case of one-dimensional Navier-Stokes system, the no-slip boundary condition is $u^\varepsilon = 0$(u^ε is the velocity of the fluid) on the boundary, which is uniformly characteristic for the corresponding Euler system. Although the significance of this type of boundary condition is clear, yet the mathematical analysis is very difficult and the progress is much limited. In contrast to the non-characteristic case, the width of the boundary layer in the characteristic case is in the order of the square root of the coefficient of viscosity, and it is well known in fluid mechanics [5] that very complicated instabilities phenomena may appear in this small zone. In particular, with the artificial viscosity, a boundary layer analysis was done for multidimensional semilinear system in [14] and for quasilinear totally characteristic systems in [11]. As a physical model with real viscosity, due to the loss of the regularity of the solution to the initial boundary value problem of the nonlinear Prandtl equation, see [35], a linearized Navier-Stokes [resp. Euler] system for the two-dimensional compressible isentropic fluid with no-slip boundary condition(characteristic boundary)is investigated by Xin and Yanagisawa [33]. In that paper, the boundary layer profile is dominated by the Prandtl equation. The authors showed the asymptotic behavior as ε tends to zero and gave a rigorous analysis of the boundary layer appearing in this process. This amounted to constructing interior and boundary layer expansions. The terms of the interior expansion arise as solutions of the corresponding (inhomogeneous) Euler equations. Then, rescaling x_1 by the order ε, the terms of the boundary expansion are derived as solutions of a family of ordinary and partial differential equations. Then by an energy estimate, the stability of the viscous boundary layer was established.

Here, we start with the following topics. As is well-known that the motion of the one-dimensional compressible viscous flow with heat-conduction without external force can be described by the following full Navier-Stokes equations

$$\begin{cases} \varrho_t + (\varrho u)_x = 0, \\ (\varrho u)_t + (\varrho u^2 + p)_x = \mu u_{xx}, \\ (\varrho E)_t + (\varrho u E + up)_x = \mu(u u_x)_x + k\theta_{xx}, \end{cases} \tag{0.0.5}$$

for $(x,t) \in \mathbb{R}^1_+ \times [0,T]$, where $\varrho, u, \theta, p(\varrho, e)$ are the density, velocity, temperature and pressure; $e(\varrho,\theta)$ represents the internal energy, and the energy $E = \frac{1}{2}u^2 + e$, see [31]. On the other hand, the motion of an inviscid compressible fluid is described by the full Euler equations, which are obtained by taking the coefficient of the viscosity μ and the coefficient of heat conductivity k in (0.0.5) to be zero:

$$\begin{cases} \varrho_t + (\varrho u)_x = 0, \\ (\varrho u)_t + (\varrho u^2 + p)_x = 0, \\ (\varrho E)_t + (\varrho u E + up)_x = 0, \end{cases} \tag{0.0.6}$$

for $(x,t) \in \mathbb{R}^1_+ \times [0,T]$. In this case, the nonslip boundary condition is given as

$$u(x=0,t) = 0. \tag{0.0.7}$$

This obviously makes the boundary be uniformly characteristic for the Euler system (0.0.6). To determine the solutions to the equations (0.0.5), we should impose a Dirichlet boundary condition on the temperature. We will formulate it in details in Chapter 1.

Since two-parameter perturbation problems are not in the scope of this thesis, for simplicity, we assume that the coefficients of viscosity and the heat conductivity are proportional to the same parameter, say, ε^2 with $0 < \varepsilon < 1$. The aim of this paper is to study the asymptotic limiting behavior of the solutions to the initial boundary value problem of the linearized Navier-Stokes equations as $\varepsilon \to 0$, for concrete form see (1.2.2)–(1.2.4) in the Chapter 1. Due to the disparity of the boundary conditions (1.1.3) and (1.1.6), the solutions to the initial boundary value problem (1.2.2)–(1.2.4) will exhibit multi-scale behavior in the uniform norm, which will lead to the phenomena of the thermal boundary layer. To do the analysis on this boundary layer phenomena, by means of the asymptotic analysis with multiple scales, for reference, see [25], [27] and [28], we combine the inner and boundary expansions to get an approximate solution of the initial boundary value problem of the linearized Navier-Stokes equations (1.2.2)–(1.2.4), which is expected to describe the boundary layer phenomena as well as possible.

Being enlightened by the work in [33], we first introduce an nonlinear transformation to change the unknown functions U into the variables V, as seen in the next chapter. Then by applying the asymptotic analysis of the inner expansion, we determine each term of the inner expansion as a solution of the inhomogeneous linearized Euler equations. This reflects the fact that the inner expansion terms describe the motion of the fluid far away from the boundary. In particular, we notice that the first-order term of the inner expansion is determined by the solution of the linearized Euler equations (1.2.10) with the same initial boundary conditions (1.2.11) and (1.2.12). Then we rescale the spatial variable x by the diffusive scale ε due to the characteristic boundary, it is seen that the two components of each order of the boundary expansion obey a family of ordinary differential equations. In order to exclude the effect of the fluid motion outside the boundary layer from the boundary expansion terms, we impose the decay conditions at infinity. The other component of each order of the boundary terms is dominated by a family of parabolic equations, we impose the inhomogeneous boundary condition of which the data consists of the same order inner expansion term in order to match the boundary condition (1.2.3), taking account of the zero initial condition, we can get the solutions of these equations. Concerning the boundary condition (1.2.11), appropriate inhomogeneous boundary data are also required for the corresponding inner expansions. Thus we can determine all the terms in the expansion. With such approximate solution, we then apply the techniques of energy methods to show the pointwise error estimates of the approximate solution. Owing to the incompleteness of the parabolicity of the Navier-Stokes equations for a compressible viscous fluid with heat conduction and the appearance of the boundary layer, the procedure of the error estimates needs careful observations of the structure between the equations and the boundary condition of the initial boundary value problem of the linearized Navier-Stokes equations. We first give the basic L^2 estimate of the error term itself (see Lemma 3.2.1 in Chapter 3). With a view to the homogenous initial and boundary condition by considering the condition (1.2.1), we then give the estimate of the tangential derivative of the error term (see Lemma 1.2.1); finally, using the equations (3.1.16)–(3.1.17) and the previous estimates, we give the L^2 estimate of the normal derivative of the error term. All these together leads to the pointwise estimate of the error term of the approximate solution , which readily yields the main result, i.e. the uniform stability result for the linearized Navier-Stokes solution in the zero-dissipation limit, as stated in Chapter 3.

This thesis is organized as follow: In Chapter 1, the initial boundary value problem of the linearized Navier-Stokes equations and the corresponding initial boundary value problem of the linearized Euler system that we treat are formulated. In Chapter 2, an approximate solution of the initial boundary value problem of the linearized Navier-Stokes equations is constructed by the method of two-scale asymptotic expansions. In Chapter 3, the error estimates of the approximate solution and the uniform stability result are given.

Chapter 1

Formulation of the Problem

In this chapter, we will illustrate how the initial boundary value problem of the linearized Navier-Stokes equations and the corresponding initial boundary value problem of the linearized Euler equations are formulated.

1.1 Reformulated Navier-Stokes Equations

The system (0.0.5) can be reduced to

$$\begin{cases} \varrho_t + (\varrho u)_x = 0, \\ \varrho u_t + \varrho u u_x + p_x = \mu u_{xx}, \\ \varrho e_t + \varrho u e_x + p u_x = \mu u_x \cdot u_x + k\theta_{xx}, \end{cases} \tag{1.1.1}$$

for $(x,t) \in \mathbb{R}^+ \times [0,T]$. We wish to conclude by giving the example of the ideal gas which obeys $p = R\varrho\theta$, and $e = c_v\theta$, where $R > 0$ is called the ideal gas constant, $c_v > 0$ represents the specific heat at constant volume and the adiabatic constant γ is setting by $\gamma = 1 + \frac{R}{c_v}$, whose interesting region for physical applications is $(1, \frac{5}{3}]$ (see [20]). So for the ideal gas, assuming $\mu = \varepsilon^2$, and $k = h\varepsilon^2$ with h a positive constant. The Navier-Stokes equations are written as

$$\begin{cases} \varrho_t + \varrho_x u + \varrho u_x = 0, \\ u_t + u u_x + \dfrac{R\theta}{\varrho} p_x + R\theta_x = \dfrac{\varepsilon^2}{\varrho} u_{xx}, \\ \theta_t + u\theta_x + \dfrac{R\theta}{c_v} u_x = \dfrac{\varepsilon^2}{\varrho c_v} u_x u_x + \dfrac{h\varepsilon^2}{\varrho c_v} \theta_{xx}. \end{cases} \tag{1.1.2}$$

for $(x,t) \in \mathbb{R}^1_+ \times [0,T]$. We impose the boundary conditions as

$$u(x=0,t) = 0, \quad and \quad \theta(x=0,t) = \theta_1 > 0, \tag{1.1.3}$$

and the initial condition

$$(\varrho, u, \theta)^T(x,0) = (\varrho_0, u_0, \theta_0)^T(x), \quad for\ x \in \mathbb{R}^+. \tag{1.1.4}$$

The associated initial-boundary value problem for Euler equations are

$$\begin{cases} \varrho_t + \varrho_x u + \varrho u_x = 0, \\ u_t + u u_x + \dfrac{R\theta}{\varrho} p_x + R\theta_x = 0, \\ \theta_t + u\theta_x + \dfrac{R\theta}{c_v} u_x = 0, \end{cases} \tag{1.1.5}$$

for $(x,t) \in \mathbb{R}^1_+ \times [0,T]$ with

$$u(x=0,t) = 0, \tag{1.1.6}$$

and the same initial condition as in (1.1.4).

Denote $U = (\varrho, u, \theta)^T$. Then the initial-boundary value problem (1.1.2)–(1.1.4) are simply described as follows

$$\begin{cases} U_t + A(U)U_x = \varepsilon^2 B(U) U_{xx} + \varepsilon^2 Q(U)(U_x, U_x), & in\ \mathbb{R}^+ \times [0,T], \\ M^+ U = \begin{pmatrix} 0 \\ \theta_1 \end{pmatrix}, & M^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ U(x,0) = (\varrho_0, u_0, \theta_0)^T(x) \equiv U_0(x), & for\ x \in \mathbb{R}^+. \end{cases} \tag{1.1.7}$$

where

$$A(U) = \begin{pmatrix} u & \varrho & 0 \\ \frac{R\theta}{\varrho} & u & R \\ 0 & \frac{R\theta}{c_v} & u \end{pmatrix},$$

$$B(U) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\varrho} & 0 \\ 0 & 0 & \frac{h}{\varrho c_v} \end{pmatrix}, \quad and \quad Q(U) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{\varrho c_v} & 0 \end{pmatrix}$$

and the initial boundary value problem of the Euler system is

$$\begin{cases} U_t^0 + A(U)U_x^0 = 0, & in\ \mathbb{R}^+ \times [0,T], \\ M^0 U^0 = 0, & M^0 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \\ U^0(x,0) = U_0(x), & for\ x \in \mathbb{R}^+. \end{cases} \tag{1.1.8}$$

It is well-known that the system (1.1.8) is strictly hyperbolic and its characteristic speeds are

$$\lambda_0 = u, \quad \lambda_1 = u + c, \quad \lambda_2 = u - c,$$

where $c = \sqrt{\gamma R\theta}$ is the sonic speed with $\gamma = 1 + \frac{R}{C_v}$(see [19]). And the corresponding left eigenvectors of $A(U)$ with respect to $\lambda_j (j = 0,1,2)$ are given by

$$l_0 = (\frac{R\theta}{\varrho}, 0, -C_v),$$
$$l_1 = (R\theta, c\varrho, R\varrho),$$
$$l_2 = (R\theta, -c\varrho, R\varrho).$$

Set

$$V = TLU = \begin{pmatrix} (R - C_v)\theta \\ 2u \\ 4R\varrho \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix},$$

where the nonlinear transformation

$$L(U) = (l_0, l_1, l_2)^T,$$

and

$$T(U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{c\varrho} & -\frac{1}{c\varrho} \\ 0 & \frac{1}{\theta} & \frac{1}{\theta} \end{pmatrix}.$$

Then in terms of V the Navier-Stokes equations can be written as

$$\partial_t V + L^\varepsilon V = 0, \quad in\ \mathbb{R}^+ \times [0,T],$$

where

$$\begin{aligned} L^\varepsilon V =& \tilde{A}(V)\partial_x V + W(V)V + \breve{A}(V)V - \varepsilon^2 \tilde{B}(V)\partial_x^2 V - 2\varepsilon^2 \breve{B}(V)\partial_x V \\ & - \varepsilon^2 \hat{B}(V)V - \varepsilon^2 \tilde{Q}(V) - 2\varepsilon^2 \breve{Q}(V) - \varepsilon^2 \hat{Q}(V). \end{aligned} \tag{1.1.9}$$

with

$$\tilde{A}(V) = TLAL^{-1}T^{-1} = \begin{pmatrix} u & 0 & 0 \\ 0 & u & -\frac{\theta}{\varrho} \\ 0 & \frac{c^2\varrho}{\theta} & u \end{pmatrix},$$

$$\begin{aligned} W(V) &= TL\partial_t(L^{-1}T^{-1}) \\ &= \begin{pmatrix} \frac{R\theta}{\varrho} & 0 & -C_v \\ 0 & 2 & 0 \\ 2R & 0 & \frac{2R\varrho}{\theta} \end{pmatrix} \partial_t \begin{pmatrix} \frac{\varrho}{\theta(R+C_v)} & 0 & \frac{C_v}{2R(R+C_v)} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{R+C_v} & 0 & \frac{\theta}{2\varrho(R+C_v)} \end{pmatrix} \\ &\triangleq \begin{pmatrix} \omega_0 & 0 & \omega_1 \\ 0 & 0 & 0 \\ \omega_2 & 0 & \omega_3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \breve{A}(V) &= TLA\partial_x(L^{-1}T^{-1}) \\ &= \begin{pmatrix} \frac{R\theta}{\varrho} & 0 & -C_v \\ 0 & 2 & 0 \\ 2R & 0 & \frac{2R\varrho}{\theta} \end{pmatrix} \begin{pmatrix} u & \varrho & 0 \\ \frac{R\theta}{\varrho} & u & R \\ 0 & \frac{R\theta}{c_v} & u \end{pmatrix} \partial_x \begin{pmatrix} \frac{\varrho}{\theta(R+C_v)} & 0 & \frac{C_v}{2R(R+C_v)} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{R+C_v} & 0 & \frac{\theta}{2\varrho(R+C_v)} \end{pmatrix} \\ &\triangleq \begin{pmatrix} \tau_0 & 0 & \tau_1 \\ \tau_2 & 0 & \tau_3 \\ \tau_4 & 0 & \tau_5 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned}
\tilde{B}(V) &= TLBL^{-1}T^{-1} \\
&= \begin{pmatrix} \frac{R\theta}{\varrho} & 0 & -C_v \\ 0 & 2 & 0 \\ 2R & 0 & \frac{2R\varrho}{\theta} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\varrho} & 0 \\ 0 & 0 & \frac{h}{\varrho c_v} \end{pmatrix} \begin{pmatrix} \frac{\varrho}{\theta(R+C_v)} & 0 & \frac{C_v}{2R(R+C_v)} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{R+C_v} & 0 & \frac{\theta}{2\varrho(R+C_v)} \end{pmatrix} \\
&= \begin{pmatrix} \frac{h}{R+C_v} & 0 & -\frac{h\theta}{2\varrho(R+C_v)} \\ 0 & \frac{1}{\varrho} & 0 \\ \frac{2hR}{C_v\theta(R+C_v)} & 0 & \frac{hR}{C_v\varrho(R+C_v)} \end{pmatrix} \triangleq \begin{pmatrix} b_0 = \frac{h}{R+C_v} & 0 & b_1 \\ 0 & b_2 & 0 \\ b_3 & 0 & b_4 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\breve{B}(V) &= TLB\partial_x(L^{-1}T^{-1}) \\
&= \begin{pmatrix} \frac{R\theta}{\varrho} & 0 & -C_v \\ 0 & 2 & 0 \\ 2R & 0 & \frac{2R\varrho}{\theta} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\varrho} & 0 \\ 0 & 0 & \frac{h}{\varrho c_v} \end{pmatrix} \partial_x \begin{pmatrix} \frac{\varrho}{\theta(R+C_v)} & 0 & \frac{C_v}{2R(R+C_v)} \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{R+C_v} & 0 & \frac{\theta}{2\varrho(R+C_v)} \end{pmatrix} \\
&\triangleq \begin{pmatrix} 0 & 0 & k_0 \\ 0 & 0 & 0 \\ 0 & 0 & k_1 \end{pmatrix},
\end{aligned}$$

$$\hat{B}(V) = TLB\partial_x^2(L^{-1}T^{-1}) \triangleq \begin{pmatrix} 0 & 0 & k_2 \\ 0 & 0 & 0 \\ 0 & 0 & k_3 \end{pmatrix}$$

and

$$\begin{aligned}
\tilde{Q}(V) &= TLQ(U)L^{-1}T^{-1}(\partial_x V, \partial_x V), \\
\breve{Q}(V) &= TLQ(U)\big(\partial_x(L^{-1}T^{-1})V, L^{-1}T^{-1}\partial_x V\big), \\
\hat{Q}(V) &= TLQ(U)\partial_x(L^{-1}T^{-1})(V, V).
\end{aligned}$$

1.2 Linearized Problems

We linearize the equations around a smooth function $V' = (v_0', v_1', v_2')^T$ that satisfies the following conditions

$$\begin{cases} V' \in C_B^\infty(\mathbb{R}^+ \times [0,T]), \\ x\partial_x V' \in C_B(\mathbb{R}^+ \times [0,T]), \\ \varrho' \geq c_0 > 0, \quad in\ \mathbb{R}^+ \times [0,T], \\ \theta' \geq c_1 > 0, \quad in\ \mathbb{R}^+ \times [0,T], \\ v_1'(x=0,t) = 0, \end{cases} \tag{1.2.1}$$

where c_0 and c_1 are constants and T is any finite positive number. Here $C_B^\infty(\mathbb{R}^+ \times [0,T])$ denotes the space of functions whose derivatives are bounded and continuous in $\mathbb{R}^+ \times [0,T]$, and $C_B(\mathbb{R}^+ \times [0,T])$ the space of bounded and continuous functions in $\mathbb{R}^+ \times [0,T]$. It should be noted that the regularity conditions in (1.2.1) is just for simplicity of presentation, it is not essential, and it will be clear from our analysis in the following chapters. Then the initial boundary value problem of the linearized Navier-Stokes equations we treat in this paper is written as

$$\partial_t V^\varepsilon + \mathcal{L}^\varepsilon V^\varepsilon + \mathcal{F}(x,t) = 0, \qquad in\ \mathbb{R}^+ \times [0,T], \tag{1.2.2}$$

$$\mathcal{M}^+ V^\varepsilon = \begin{pmatrix} (R - C_v)\theta_1 \\ 0 \end{pmatrix}, \qquad \mathcal{M}^+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{1.2.3}$$

$$V^\varepsilon(x,0) = TLU_0 \triangleq V_0(x), \qquad for\ x \in \mathbb{R}^+. \tag{1.2.4}$$

where

$$\begin{aligned} \mathcal{L}^\varepsilon V =& \tilde{\mathcal{A}}(x,t)\partial_x V + \mathcal{W}(x,t)V + \breve{\mathcal{A}}(x,t)V \\ & - \varepsilon^2 \tilde{\mathcal{B}}(x,t)\partial_x^2 V - 2\varepsilon^2 \breve{\mathcal{B}}(x,t)\partial_x V - \varepsilon^2 \hat{\mathcal{B}}(x,t)V, \end{aligned} \tag{1.2.5}$$

and

$$\mathcal{F}(x,t) = \varepsilon^2 S^0(x,t) + \varepsilon^2 S^1(x,t) + \varepsilon^2 S^2(x,t).$$

where $\tilde{\mathcal{A}}(x,t)$ is the linearization of $\tilde{A}(V)$ around V', i.e.

$$\tilde{\mathcal{A}}(x,t) = \begin{pmatrix} u' & 0 & 0 \\ 0 & u' & -\frac{\theta'}{\varrho'} \\ 0 & \frac{c'^2\varrho'}{\theta'} & u' \end{pmatrix},$$

the meanings of $\mathcal{W}(x,t)$, $\breve{\mathcal{A}}(x,t)$, $\hat{\mathcal{B}}(x,t)$ are similar. We should point out that

$$S^0(x,t) = -\tilde{Q}(V') - \breve{Q}(V') - \hat{Q}(V'),$$

and $\tilde{\mathcal{B}}(x,t)$ is the following linearized $\tilde{B}(x,t)$,

$$\tilde{\mathcal{B}}(x,t) = \begin{pmatrix} b_0 & 0 & 0 \\ 0 & b_2' & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.2.6}$$

with

$$S^1(x,t) = \begin{pmatrix} b_1'\partial_x^2 v_2' \\ 0 \\ b_3'\partial_x^2 v_0' + b_4'\partial_x^2 v_2' \end{pmatrix}, \tag{1.2.7}$$

and $\breve{\mathcal{B}}(x,t)$ is the linearized $\breve{B}(x,t)$,

$$\breve{\mathcal{B}}(x,t) = \begin{pmatrix} 0 & 0 & k_0' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.2.8}$$

with

$$S^2(x,t) = \begin{pmatrix} 0 \\ 0 \\ k_1'\partial_x v_2' \end{pmatrix}. \tag{1.2.9}$$

From now on we omit the prime for simplicity. The corresponding initial boundary value problems of the linearized Euler equations are

$$\partial_t V^0 + \mathcal{L}^0 V^0 = 0, \qquad in\ \mathbb{R}^1_+ \times [0,T], \tag{1.2.10}$$

$$M^0 V^\varepsilon = 0, \qquad M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.2.11}$$

$$V^0(x,0) = V_0(x), \qquad for\ x \in \mathbb{R}^+. \tag{1.2.12}$$

where

$$\mathcal{L}^0 V = \tilde{\mathcal{A}}(x,t)\partial_x V + \mathcal{W}(x,t)V + \breve{\mathcal{A}}(x,t)V. \tag{1.2.13}$$

It is well-known that in order to obtain the smooth solution to (1.2.2)–(1.2.4), it is necessary for us to impose some compatibility conditions on the initial and boundary values of the solutions. We define inductively the p-Cauchy data ($p = 0, 1, 2, \cdots$) of (1.2.2)–(1.2.4) by

$$V^\varepsilon(x,0) = V_0(x), \tag{1.2.14}$$

$$\begin{aligned}\partial_t^p V^\varepsilon(x,0) = &\sum_{s=0}^{p-1} \begin{pmatrix} p-1 \\ s \end{pmatrix} \Big\{ -(\partial_t^s \tilde{A} \partial_t^{p-1-s}(\partial_x V^\varepsilon))(x,0) \\ &- (\partial_t^s \mathcal{W} + \breve{A}\partial_t^{p-1-s} V)(x,0) + \varepsilon^2(\partial_t^s \bar{B}\partial_t^{p-1-s})(\partial_x V^\varepsilon)(x,0) \\ &+ \varepsilon^2(\partial_t^s \hat{B}\partial_t^{p-1-s} V^\varepsilon)(x,0)\Big\} + \varepsilon^2(\partial_t^p(S^0 + S^1 + S^2))(x,0).\end{aligned} \tag{1.2.15}$$

Then the initial data $V_0(x)$ of (1.2.4) is said to satisfy the compatibility condition of order m for the initial boundary value problem (1.2.2)–(1.2.4) for any $\varepsilon > 0$ if

$$\mathcal{M}^+ \partial_t^p V^\varepsilon(0,0) = \begin{cases} ((R - C_v)\theta_1, 0)^T, & for\ p = 0, \\ 0, & p = 1, 2, \cdots, m. \end{cases} \tag{1.2.16}$$

And the compatibility condition of order m of the initial boundary value problem for the linearized Euler equations is

$$M^0 \partial_t^p V^0(0,0) = 0, \quad for\ p = 0, 1, 2, \cdots, m. \tag{1.2.17}$$

To isolate the effects of the boundaries, we will consider the smooth viscous flows for the initial boundary value problem (1.2.2)–(1.2.4) of the linearized Navier-Stokes equations of a compressible viscous fluid for fixed ε, and this is guaranteed by the following local existence theorem.

Proposition 1.2.1 *(Local Existence) Let $\varepsilon > 0$ be constant. Let $m \geq 2$ be integer. Suppose that the initial data $V_0 \in H^m(\mathbb{R}^+)$ satisfies the compatibility* *condition of order* $\left[\frac{m}{2}\right]-1$ *for the initial boundary value problem* (1.2.2)-(1.2.4). *Then there exists a unique solution* V^ε *of* (1.2.2)-(1.2.4) *such that*

$$V^\varepsilon(x,t) \in \bigcap_{j=0}^{\left[\frac{m}{2}\right]-1} C^j([0,T]; H^{m-2j}(\mathbb{R}^+)). \tag{1.2.18}$$

The proof of the Proposition 1.2.1 is a standard procedure, which is a slight modification of the argument by Matsumura, A. and Nishida, T. [21],[22].

Chapter 2

Construction of the Approximate Solution

In this chapter, we discuss how the linearized Euler equations can be formally derived from the linearized Navier-Stokes equations through different scaling and asymptotic expansions. The desired approximate viscous solutions to (1.2.2)–(1.2.4) which approximates the given smooth solution of the linearized Euler equations uniformly away from the boundary and possess a sharp change near the boundary. The introduction of two different scalings, typical in perturbation theory [16], is formally necessary to describe two different regions of the flow: the inviscid region (far away from the boundary) and the viscous region (close to the boundary) where the viscous forces can not be neglected even for small viscosity.

2.1 Two-scale Asymptotic Expansions

We approximate the viscous solutions of (1.2.2)–(1.2.4) uniformly up to the boundary by the following two-scale expansions

$$V^{app} = V_{in}(\varepsilon, x, t) + V_{bd}(\varepsilon, \frac{x}{\varepsilon}, t), \tag{2.1.1}$$

where the inner expansion V_{in} and the boundary expansion V_{bd} are given, respectively, as the truncated regular series

$$V_{in}(\varepsilon,x,t) = a^0(x,t) + \varepsilon a^1(x,t) + \varepsilon^2 a^2(x,t), \tag{2.1.2}$$

and for $y = \dfrac{x}{\varepsilon}$,

$$V_{bd}(\varepsilon,\frac{x}{\varepsilon},t) = B_0(y,t) + \varepsilon B^1(y,t) + \varepsilon^2 B^2(y,t), \tag{2.1.3}$$

here $a^i = (a_0^i, a_1^i, a_2^i)^T$ and $B^i = (B_0^i, B_1^i, B_2^i)^T$, $i = 0,1,2$ are all vector-valued functions to be determined. Then we give a formal expansion of $\partial_t V_{in} + \mathcal{L}^\varepsilon V_{in}$ and $\partial_t V_{bd} + \mathcal{L}^\varepsilon V_{bd}$ in terms of ε.

$$\begin{aligned}
&\partial_t V_{in} + \mathcal{L}^\varepsilon V_{in} + \mathcal{F}(x,t)\\
=&\Big(\partial_t a^0 + \tilde{\mathcal{A}}(x,t)\partial_x a^0 + \mathcal{W}(x,t)a^0 + \breve{\mathcal{A}}(x,t)a^0\Big)\\
&+\varepsilon\Big(\partial_t a^1 + \tilde{\mathcal{A}}(x,t)\partial_x a^1 + \mathcal{W}(x,t)a^1 + \breve{\mathcal{A}}(x,t)a^1\Big)\\
&+\varepsilon^2\Big(\partial_t a^2 + \tilde{\mathcal{A}}(x,t)\partial_x a^2 + \mathcal{W}(x,t)a^2 + \breve{\mathcal{A}}(x,t)a^2\\
&-\tilde{\mathcal{B}}(x,t)\partial_x^2 a^0 - 2\breve{\mathcal{B}}(x,t)\partial_x a^0 - \hat{\mathcal{B}}(x,t)a^0 + S^0(x,t) + S^1(x,t) + S^2(x,t)\Big)\\
&+\varepsilon^3\Big(-\tilde{\mathcal{B}}(x,t)\partial_x^2 a^1 - 2\breve{\mathcal{B}}(x,t)\partial_x a^1 - \hat{\mathcal{B}}(x,t)a^1\Big)\\
&+\varepsilon^4\Big(-\partial_x^2 a^2 - 2\breve{\mathcal{B}}(x,t)\partial_x a^2 - \hat{\mathcal{B}}(x,t)a^2\Big)
\end{aligned}\tag{2.1.4}$$

and

$$\partial_t V_{bd} + \mathcal{L}^\varepsilon V_{bd} + \mathcal{F}(x,t) = \sum_{i=1}^{7} \varepsilon^{i-2} K^i(t), \tag{2.1.5}$$

where the terms K^i, $i = 0,1,\cdots,7$, are given as follows

$$K^1(t) = \tilde{\mathcal{A}}(0,t)\partial_y B^0, \tag{2.1.6}$$

$$\begin{aligned}
K^2(t) =&\partial_t B^0 + y\partial_x\tilde{\mathcal{A}}(0,t)\partial_y B^0 + \tilde{\mathcal{A}}(0,t)\partial_y B^1\\
&+\mathcal{W}(0,t)B^0 + \breve{\mathcal{A}}(0,t)B^0 - \tilde{\mathcal{B}}(0,t)\partial_y^2 B^0,
\end{aligned}\tag{2.1.7}$$

$$
\begin{aligned}
K^3(t) =&\partial_t B^1 + \frac{1}{2}y^2\partial_x^2\tilde{\mathcal{A}}(0,t)\partial_y B^0 + y\partial_x\tilde{\mathcal{A}}(0,t)\partial_y B^1 + \tilde{\mathcal{A}}(0,t)\partial_y B^2\\
&+ y\partial_x\mathcal{W}(0,t)B^0 + \mathcal{W}(0,t)B^1 + y\partial_x\breve{\mathcal{A}}(0,t)B^0 + \breve{\mathcal{A}}(0,t)B^1\\
&- y\partial_x\tilde{\mathcal{B}}(0,t)\partial_y^2 B^0 - \tilde{\mathcal{B}}(0,t)\partial_y^2 B^1 - 2\breve{\mathcal{B}}(0,t)\partial_y B^0,
\end{aligned}
\tag{2.1.8}
$$

$$
\begin{aligned}
K^4(t) =&\partial_t B^2 + \frac{1}{6}y^3\partial_x^3\tilde{\mathcal{A}}(0,t)\partial_y B^0 + \frac{1}{2}y^2\partial_x^2\tilde{\mathcal{A}}(0,t)\partial_y B^1 + y\partial_x\tilde{\mathcal{A}}(0,t)\partial_y B^2\\
&+ \frac{1}{2}y^2\partial_x^2\mathcal{W}(0,t)B^0 + y\partial_x\mathcal{W}(0,t)B^1 + \mathcal{W}(0,t)B^2 + \frac{1}{2}y^2\partial_x^2\breve{\mathcal{A}}(0,t)B^0\\
&+ y\partial_x\breve{\mathcal{A}}(0,t)B^1 + \breve{\mathcal{A}}(0,t)B^2 - \frac{1}{2}y^2\partial_x^2\tilde{\mathcal{B}}(0,t)\partial_y^2 B^0 - y\partial_x\tilde{\mathcal{B}}(0,t)\partial_y^2 B^1\\
&- \tilde{\mathcal{B}}(0,t)\partial_y^2 B^2 - 2y\partial_x\breve{\mathcal{B}}(0,t)\partial_y B^0 - 2\breve{\mathcal{B}}(0,t)\partial_y B^1 - \hat{\mathcal{B}}(0,t)B^0\\
&+ S^0(0,t) + S^1(0,t) + S^2(0,t),
\end{aligned}
\tag{2.1.9}
$$

$$
\begin{aligned}
K^5(t) =&(\partial_x^3\tilde{\mathcal{A}})_R(\varepsilon y,t)y^4\partial_y B^0 + (\partial_x^2\tilde{\mathcal{A}})_R(\varepsilon y,t)y^3\partial_y B^1 + (\partial_x^1\tilde{\mathcal{A}})_R(\varepsilon y,t)y^2\partial_y B^2\\
&+ (\partial_x^2\mathcal{W})_R(\varepsilon y,t)y^3B^0 + (\partial_x^1\mathcal{W})_R(\varepsilon y,t)y^2B^1 + (\partial_x^0\mathcal{W})_R(\varepsilon y,t)yB^2\\
&+ (\partial_x^2\breve{\mathcal{A}})_R(\varepsilon y,t)y^3B^0 + (\partial_x^1\breve{\mathcal{A}})_R(\varepsilon y,t)y^2B^1 + (\partial_x^0\breve{\mathcal{A}})_R(\varepsilon y,t)yB^2\\
&- (\partial_x^2\tilde{\mathcal{B}})_R(\varepsilon y,t)y^3\partial_y^2 B^0 + (\partial_x^1\tilde{\mathcal{B}})_R(\varepsilon y,t)y^2\partial_y^2 B^1 + (\partial_x^0\tilde{\mathcal{B}})_R(\varepsilon y,t)y\partial_y^2 B^2\\
&- 2(\partial_x^1\breve{\mathcal{B}})_R(\varepsilon y,t)y^2\partial_y B^0 - 2(\partial_x^0\breve{\mathcal{B}})_R(\varepsilon y,t)y\partial_y B^1 - 2\breve{\mathcal{B}}(0,t)\partial_y B^2\\
&- (\partial_x^0\hat{\mathcal{B}})_R(\varepsilon y,t)yB^0 - \hat{\mathcal{B}}(0,t)B^1 + y(\partial_x^0 S^0)_R(\varepsilon y,t)\\
&+ y(\partial_x^0 S^1)_R(\varepsilon y,t) + y(\partial_x^0 S^2)_R(\varepsilon y,t),
\end{aligned}
\tag{2.1.10}
$$

$$
K^6(t) = -2(\partial_x^0\breve{\mathcal{B}})_R(\varepsilon y,t)y\partial_y B^2 - (\partial_x^0\hat{\mathcal{B}})_R(\varepsilon y,t)yB^1 - \hat{\mathcal{B}}(0,t)B^2, \tag{2.1.11}
$$

and

$$
K^7(t) = -(\partial_x^0\hat{\mathcal{B}})_R(\varepsilon y,t)yB^2. \tag{2.1.12}
$$

Here we use the notation that, for a smooth matrix-valued function $\mathcal{A}(x,t) = \mathcal{A}(\varepsilon y,t)$,

$$
(\partial_x^k\mathcal{A})_R(\varepsilon y,t) \triangleq \frac{1}{k!}\int_0^1 \frac{\partial^{k+1}}{\partial x^{k+1}}\mathcal{A}(\varepsilon y\xi,t)(1-\xi)^k d\xi. \tag{2.1.13}
$$

2.2 Determination of Each Inner and Boundary Terms

We now detail the construction of the various orders of inner and boundary expansions. It should be noted that the same order of inner and boundary functions will be constructed simultaneously due to their coupling at the boundary. To determine the solutions of the governing problems of the terms of the inner expansion, we should first introduce here the existence theorem for the following general problem:

$$\partial_t a^i + \mathcal{L}^0 a^i = F(x,t), \qquad in\ \mathbb{R}^+ \times [0,T], \tag{2.2.1}$$

$$M^0 a^i(x,0) = 0, \qquad for\ t \in [0,T], \tag{2.2.2}$$

$$a^i(x,0) = f(x), \qquad for\ x \in \mathbb{R}^+. \tag{2.2.3}$$

Proposition 2.2.1 *Let $k \geq 1$ be an integer. Suppose that $f(x) \in H^k(\mathbb{R}^+)$ and $F(x,t) \in H^k(\mathbb{R}^+ \times [0,T])$ satisfy the compatibility condition of order $k-1$ for (2.2.1)-(2.2.3). Then there exists a unique solution a^i of (2.2.1)-(2.2.3), such that*

$$a^i \in \bigcap_{j=0}^{k} C^j\big([0,T]; H^{k-j}(\mathbb{R}^+)\big).$$

Proof: Introduce

$$\mathcal{A}_0(x,t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\varrho}{\theta} & 0 \\ 0 & 0 & \frac{\theta}{c^2\varrho} \end{pmatrix} \triangleq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \delta \end{pmatrix}, \tag{2.2.4}$$

where β and δ are functions of x and t. $\mathcal{A}_0$ is then positive definite due to the positivity and continuity of V' as seen by (1.2.1). Noticing both

$$\mathcal{A}_0(x,t)\tilde{A}(x,t) = \begin{pmatrix} u & 0 & 0 \\ 0 & \frac{\varrho u}{\theta} & 1 \\ 0 & 1 & \frac{\theta u}{c^2\varrho} \end{pmatrix}$$

and A_0 are symmetric and smooth, we multiply the matrix $\mathcal{A}_0$ on both sides of the equation in (2.2.1) and get the following initial boundary value problem for the symmetric first order hyperbolic system

$$\begin{cases} \mathcal{A}_0\partial_t a^i + \bar{\mathcal{L}}^0 a^i = F(x,t), & in\ \mathbb{R}^+ \times [0,T], \\ M^0 a^i(0,t) = 0, & for\ t \in [0,T], \\ a^i(x,0) = f(x), & for\ x \in \mathbb{R}^+. \end{cases} \tag{2.2.5}$$

where $\bar{\mathcal{L}}^0 \triangleq \mathcal{A}_0\mathcal{L}^0$, with $\mathcal{L}^0$ the same definition as in (1.2.13) and $M^0 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$. The existence of the solutions of (2.2.5) is followed by the argument in [33], that is, there is a solution $a^i \in \bigcap_{j=0}^k C^j\big([0,T]; H^{k-j}(I)\big)$ to (2.2.5), and hence the proposition is proved. □

By setting the $\mathcal{O}(1)$-order term in (2.1.4) to be zero and imposing the boundary and initial conditions (1.2.11)–(1.2.12), we have the following initial boundary value problem of a^0,

$$\begin{cases} \partial_t a^0 + \mathcal{L}^0 a^0 = 0, & in\ \mathbb{R}^+ times[0,T], \\ M^0 a^0(0,t) = 0, & for\ t \in [0,T], \\ a^0(x,0) = V_0(x), & for\ x \in \mathbb{R}^+. \end{cases} \tag{2.2.6}$$

Then from the Proposition 2.2.1, there exists a unique solution a^0 of the problem (2.2.6) such that

$$a^0 \in \bigcap_{j=0}^{m} C^j\big([0,T]; H^{m-j}(\mathbb{R}^+)\big), \tag{2.2.7}$$

for $V_0(x) \in H^m(\mathbb{R}^+)$ and satisfies the compatibility condition of order $m-1$ for (2.2.5) with $F = 0$, and $f = V_0$. It can be seen that a^0 is a solution of the initial boundary value problem of the linearized Euler equations (1.2.10)–(1.2.12).

Next we turn to the first-order term B^0 in the boundary expansion. Setting the $\mathcal{O}(\varepsilon^{-1})$-order term in (2.1.5) to zero gives

$$K^1(t) = \tilde{A}(0,t)\partial_y B^0(y,t) = 0, \tag{2.2.8}$$

i.e.

$$\begin{pmatrix} 0 & \beta(0,t) \\ \delta(0,t) & 0 \end{pmatrix} \begin{pmatrix} \partial_y B_1^0(y,t) \\ \partial_y B_1^1(y,t) \end{pmatrix} = 0. \tag{2.2.9}$$

Since the boundary expansion V_{bd} is expected to approximate well the deviation of the solution V^ε from the inviscid flow in the vicinity of the boundary, we impose the decay condition at infinity to B^0, that is

$$B_j^0(y,t) \to 0, \quad as\ y \to +\infty, \quad j = 1,2. \tag{2.2.10}$$

The only solution to (2.2.9)–(2.2.10) is

$$B_j^0(y,t) \equiv 0, \quad y \in \mathbb{R}^+, \quad j = 1,2, \tag{2.2.11}$$

due to the condition (1.2.1).

The first component B_0^0 and the two components B_1^1 and B_2^1 are simultaneously determined by setting the $\mathcal{O}(1)$-order term in (2.1.5), i.e. $K^2(t)$ to zero, which gives

$$\begin{aligned} &\begin{pmatrix} \partial_t B_0^0(y,t) \\ \partial_t B_1^0(y,t) \\ \partial_t B_2^0(y,t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \partial_x \beta(0,t) \\ 0 & \partial_x \delta(0,t) & 0 \end{pmatrix} \begin{pmatrix} \partial_y B_0^0(y,t) \\ \partial_y B_1^0(y,t) \\ \partial_y B_2^0(y,t) \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \beta(0,t) \\ 0 & \delta(0,t) & 0 \end{pmatrix} \begin{pmatrix} \partial_y B_0^1(y,t) \\ \partial_y B_1^1(y,t) \\ \partial_y B_2^1(y,t) \end{pmatrix} + \begin{pmatrix} \omega_0 & 0 & \omega_1 \\ 0 & 0 & 0 \\ \omega_2 & 0 & \omega_3 \end{pmatrix} \begin{pmatrix} B_0^0(y,t) \\ B_1^0(y,t) \\ B_2^0(y,t) \end{pmatrix} \\ &+ \begin{pmatrix} \tau_0 & 0 & \tau_1 \\ \tau_2 & 0 & \tau_3 \\ \tau_4 & 0 & \tau_5 \end{pmatrix} \begin{pmatrix} B_0^0(y,t) \\ B_1^0(y,t) \\ B_2^0(y,t) \end{pmatrix} - \begin{pmatrix} b_0 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_y^2 B_0^0(y,t) \\ \partial_y^2 B_1^0(y,t) \\ \partial_y^2 B_2^0(y,t) \end{pmatrix} = 0. \end{aligned} \tag{2.2.12}$$

From the first equation, we get

$$\begin{cases} \partial_t B_0^0 - b_0 \partial_y^2 B_0^0 + (\omega_0 + \tau_0) B_0^0 = 0, & in\ \mathbb{R}^+ \times [0,T], \\ B_0^0(0,t) = -a_0^0(0,t) + (R - C_v)\theta_1, & for\ t \in [0,T], \\ B_0^0(y,0) = 0, & for\ y \in \mathbb{R}^+. \end{cases} \tag{2.2.13}$$

Notice that

$$\mathcal{M}^{+}(a^0+B^0)(0,t)=\big((R-C_v)\theta_1,0\big)^T,\quad for\ t\in[0,T]. \tag{2.2.14}$$

Since $b_0=\frac{h}{R+C_v}>0$, the above equation of B_0^0 is a linear uniform parabolic equation, and $B_0^0(0,t)$, $B_0^0(y,0)$ satisfy the compatibility conditions at $(0,0)$, then the existence, uniqueness is followed by the theory of linear parabolic equations, see Ladyzhenskaya [18]. Retracing the similar argument in [33] with suitable modification, we have the following regularity of B_0^0

$$\begin{gathered}\langle y\rangle^l\partial_t^k\partial_y^\alpha B_0^0(y,t)\in C^0([0,T];L^2(\mathbb{R}^+)), \qquad (2.2.15)\\ for\ k{+}\alpha\le m,\ k+\left[\frac{\alpha+1}{2}\right]\le\left[\frac{m}{2}\right],\ l\in\mathbb{N}_0,\end{gathered}$$

and

$$\partial_t^k B_0^0(y,0)=0,\quad k=0,1,\cdots,\left[\frac{m}{2}\right],\ for\ y\in\mathbb{R}^+. \tag{2.2.16}$$

where $\langle y\rangle=\sqrt{1+y^2}$, and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. And then from the second and third equations with B_0^0 is determined above, the equations for B_1^1 and B_2^1 are given by the following ODEs:

$$\partial_y\begin{pmatrix}B_1^1\\ B_2^1\end{pmatrix}(y,t)=\begin{pmatrix}H_1^1(B^0)\\ H_2^1(B^0)\end{pmatrix}(y,t),\quad for\ y\in\mathbb{R}^+, \tag{2.2.17}$$

where

$$\begin{pmatrix}H_1^1(B^0)\\ H_2^1(B^0)\end{pmatrix}(y,t)=\begin{pmatrix}-\dfrac{\tau_2}{\frac{\theta}{\varrho}(0,t)}B_0^0\\ -\dfrac{\omega_2+\tau_4}{\frac{c^2\varrho}{\theta}(0,t)}B_0^0\end{pmatrix}(y,t), \tag{2.2.18}$$

with the asymptotic behavior at infinity given as

$$B_j^1(y,t)\to 0,\quad y\to+\infty,\ j=1,2. \tag{2.2.19}$$

The solutions $B_j^1(j=1,2)$ to (2.2.17)–(2.2.19) are uniquely given by

$$B_j^1(y,t)=\int_y^\infty H_j^1(B^0)(\xi,t)d\xi,\ j=1,2. \tag{2.2.20}$$

It follows from (2.2.15) and (1.2.1) that the integral in (2.2.20) is well-defined and

$$\langle y\rangle^l \partial_t^k \partial_y^\alpha B_j^1(y,t) \in C^0([0,T]; L^2(\mathbb{R}^+)),\ j=1,2, \quad (2.2.21)$$
$$for\ k+\alpha \le m+1,\ k+\left[\frac{\alpha+1}{2}\right] \le \left[\frac{m}{2}\right],\ l\in\mathbb{N}_0.$$

Moreover, (2.2.16) shows that

$$\partial_t^k B_j^1(y,0)=0, \quad j=1,2,\ k=0,1,\cdots,\left[\frac{m}{2}\right],\ for\ y\in\mathbb{R}^+. \quad (2.2.22)$$

Next, we turn to the second-order term a^1 of the inner expansion. By setting the $\mathcal{O}(\varepsilon)$-order term in (2.1.4) to zero, we get the following initial boundary value problem for a^1:

$$\begin{cases} \partial_t a^1 + \mathcal{L}^0 a^1 = 0, & in\ \mathbb{R}^+\times[0,T], \\ M^0 a^1 = -M^0 B^1, & for\ t\in[0,T], \\ a^1(x,0)=0, & for\ x\in\mathbb{R}^+. \end{cases} \quad (2.2.23)$$

Note that $M^0 B^1$ does not contain B_0^1 and is thus a known function. Set $\tilde{a}^1 = a^1 + \bar{B}^1$, where $\bar{B}^1 = e^{-x^2}(0, B_1^1, 0)^T(0,t)$. It follows from (2.2.22) with $k=0$ that $\tilde{a}^1$ solves

$$\begin{cases} \partial_t \tilde{a}^1 + \mathcal{L}^0 \tilde{a}^1 = F(\bar{B}^1), & in\ \mathbb{R}^+\times[0,T], \\ M^0 \tilde{a}^1 = 0, & for\ t\in[0,T], \\ \tilde{a}^1(x,0)=0, & for\ x\in\mathbb{R}^+. \end{cases} \quad (2.2.24)$$

where

$$F(\bar{B}^1) = \partial_t \bar{B}^1 + \mathcal{L}^0 \bar{B}^1.$$

It follow form (1.2.1), (2.2.21) and (2.2.22) that

$$F(\bar{B}^1) \in \bigcap_{j=0}^{[\frac{m}{2}]-2} C^j([0,T]; H^{[\frac{m}{2}]-2-j}(\mathbb{R}^+)) \subset H^{[\frac{m}{2}]-2}(\mathbb{R}^+\times[0,T])$$

and

$$\partial_t^k F(\bar{B}^1)(x,0) = 0, \quad k = 0, 1, \cdots, \left[\frac{m}{2}\right] - 2, \ for\ x \in \mathbb{R}^+.$$

Since $f = 0$ and $F = F(\bar{B}^1)$ satisfy the compatibility condition of order $[\frac{m}{2}] - 3$, by Proposition 2.2.1 we know that there exists a unique solution

$$\tilde{a}^1 \in \bigcap_{j=0}^{[\frac{m}{2}]-2} C^j([0,T]; H^{[\frac{m}{2}]-2-j}(\mathbb{R}^+))$$

to (2.2.24). This implies that there exists a unique solution a^1 to (2.2.23) such that

$$a^1 \in \bigcap_{j=0}^{[\frac{m}{2}]-2} C^j([0,T]; H^{[\frac{m}{2}]-2-j}(\mathbb{R}^+)). \tag{2.2.25}$$

Then we come to determine B_0^1 and $B_j^2 (j = 1, 2)$. Requiring the $\mathcal{O}(\varepsilon)$-order term in (2.1.5) to vanish, i.e. $K^3(t) = 0$, one finds

$$\begin{aligned}
&\begin{pmatrix} \partial_t B_0^1 \\ \partial_t B_1^1 \\ \partial_t B_2^1 \end{pmatrix} + \frac{1}{2}y^2 \begin{pmatrix} \partial_x^2 u(0,t) & 0 & 0 \\ 0 & \partial_x^2 u(0,t) & \partial_x^2(\frac{\theta}{\varrho})(0,t) \\ 0 & \partial_x^2(\frac{c^2\varrho}{\theta})(0,t) & \partial_x^2 u(0,t) \end{pmatrix} \begin{pmatrix} \partial_y B_0^0 \\ \partial_y B_1^0 \\ \partial_y B_2^0 \end{pmatrix} \\
&+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\theta}{\varrho}(0,t) \\ 0 & \frac{c^2\varrho}{\theta}(0,t) & 0 \end{pmatrix} \begin{pmatrix} \partial_y^2 B_0^2 \\ \partial_y^2 B_1^2 \\ \partial_y^2 B_2^2 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 2k_0(0,t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_y B_0^0 \\ \partial_y B_1^0 \\ \partial_y B_2^0 \end{pmatrix} \\
&+ y \begin{pmatrix} \partial_x u(0,t) & 0 & 0 \\ 0 & \partial_x u(0,t) & \partial_x(\frac{\theta}{\varrho})(0,t) \\ 0 & \partial_x(\frac{c^2\varrho}{\theta})(0,t) & \partial_x u(0,t) \end{pmatrix} \begin{pmatrix} \partial_y B_0^1 \\ \partial_y B_1^1 \\ \partial_y B_2^1 \end{pmatrix} \\
&+ y \begin{pmatrix} \partial_x(\omega_0 + \tau_0)(0,t) & 0 & \partial_x(\omega_1 + \tau_1)(0,t) \\ \partial_x \tau_2(0,t) & 0 & \partial_x \tau_3(0,t) \\ \partial_x(\omega_2 + \tau_4)(0,t) & 0 & \partial_x(\omega_3 + \tau_5)(0,t) \end{pmatrix} \begin{pmatrix} B_0^0 \\ B_1^0 \\ B_2^0 \end{pmatrix} \\
&- y \begin{pmatrix} 0 & 0 & 0 \\ 0 & \partial_x b_2(0,t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_y^2 B_0^0 \\ \partial_y^2 B_1^0 \\ \partial_y^2 B_2^0 \end{pmatrix} - \begin{pmatrix} b_0 & 0 & 0 \\ 0 & b_2(0,t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_y^2 B_0^1 \\ \partial_y^2 B_1^1 \\ \partial_y^2 B_2^1 \end{pmatrix}
\end{aligned}$$

$$+\begin{pmatrix} (\omega_0+\tau_0)(0,t) & 0 & (\omega_1+\tau_1)(0,t) \\ \tau_2(0,t) & 0 & \tau_3(0,t) \\ (\omega_2+\tau_4)(0,t) & 0 & (\omega_3+\tau_5)(0,t) \end{pmatrix} \begin{pmatrix} B_0^1 \\ B_1^1 \\ B_2^1 \end{pmatrix} = 0, \tag{2.2.26}$$

from which we get the following linear parabolic equation for B_0^1,

$$\begin{aligned} &\partial_t B_0^1 - b_0 \partial_y^2 B_0^1 + y\partial_x u(0,t)\partial_y B_0^1 + (\omega_0+\tau_0)(0,t)B_0^1 \\ &+ \frac{1}{2}y^2\partial_x^2 u(0,t)\partial_y B_0^0 + y\partial_x(\omega_0+\tau_0)(0,t)B_0^0 + (\omega_1+\tau_1)(0,t)B_2^1 = 0, \end{aligned} \tag{2.2.27}$$

with the boundary condition

$$B_0^1(0,t) = -a_0^1(0,t), \tag{2.2.28}$$

and the initial condition

$$B_0^1(y,0) = 0. \tag{2.2.29}$$

Notice that

$$\mathcal{M}^+(a^1+B^1)(0,t) = 0, \quad for\ t \in [0,T]. \tag{2.2.30}$$

It follows from $b_0 > 0$ and the argument in [33], there exists a unique solution B_0^1 to (2.2.27)–(2.2.29) such that

$$\langle y\rangle^l \partial_t^k \partial_y^\alpha B_0^1 \in C^0([0,T]; L^2(\mathbb{R}^+)), \tag{2.2.31}$$

$$for\ k+\alpha \le \left[\frac{m}{2}\right] - 3,\ l \in \mathbb{N}_0,$$

and

$$\partial_t^k B_0^1(y,0) = 0, \quad k = 0,1,\cdots,\left[\frac{m}{2}\right]-3,\ for\ y \in \mathbb{R}^+. \tag{2.2.32}$$

Likewise, B_1^2 and B_2^2 satisfy

$$\partial_y \begin{pmatrix} B_1^2 \\ B_2^2 \end{pmatrix} = \begin{pmatrix} H_1^2(B^0,B^1) \\ H_2^2(B^0,B^1) \end{pmatrix}, \tag{2.2.33}$$

where

$$H_1^2(B^0,B^1) = -\left[\frac{c^2\varrho}{\theta}(0,t)\right]^{-1}\Big\{\partial_t B_2^1 + y\partial_x\left(\frac{c^2\varrho}{\theta}\right)(0,t)\partial_y B_1^1 + y\partial_x u(0,t)\partial_y B_2^1$$

$$+ y\partial_x\omega_2(0,t)B_0^0 + \omega_2(0,t)B_0^1 + \omega_3(0,t)B_2^1 + y\partial_x\tau_4(0,t)B_0^0$$
$$+ \tau_4(0,t)B_0^1 + \tau_5(0,t)B_2^1\Big\}, \tag{2.2.34}$$

and

$$H_2^2(B^0,B^1) = -[\frac{\theta}{\varrho}(0,t)]^{-1}\Big\{\partial_t B_1^1 + y\partial_x u(0,t)\partial_y B_1^1 + \partial_x\left(\frac{\theta}{\varrho}\right)(0,t)$$
$$+ y\partial_x\tau_2(0,t)B_0^0 + \tau_2(0,t)B_0^1 + \tau_3(0,t)B_2^1 - b_2(0,t)\partial_y^2 B_1^1\Big\}. \tag{2.2.35}$$

We impose also the decay conditions at infinity

$$B_j^2(y,t) \to 0, \quad as\ y \to \infty,\ j = 1,2. \tag{2.2.36}$$

The unique solution to (2.2.33)–(2.2.36) is given by

$$B_j^2(y,t) = \int_y^\infty H_j^2(B^0,B^1)(\xi,t)d\xi, \quad j = 1,2, \tag{2.2.37}$$

which satisfy

$$\langle y\rangle^l\partial_t^k\partial_y^\alpha B_j^2 \in C^0([0,T];L^2(\mathbb{R}^+)), \quad j = 1,2, \tag{2.2.38}$$
$$for\ k+\alpha \le \left[\frac{m}{2}\right] - 3,\ l \in \mathbb{N}_0,$$

Furthermore, from (2.2.16), (2.2.22) and (2.2.32), we find that

$$\partial_t^k B_j^2(y,0) = 0, \quad k = 0,1,\cdots,\left[\frac{m}{2}\right] - 3,\ j = 1,2,\ y \in \mathbb{R}^+. \tag{2.2.39}$$

By setting the $\mathcal{O}(\varepsilon^2)$-order term to zero, we have the following initial boundary value problem for a^2,

$$\begin{cases} \partial_t a^2 + \mathcal{L}^0 a^2 = \mathcal{G}(a^0), & in\ \mathbb{R}^+ \times [0,T], \\ M^0 a^2 = -M^0 B^2, & for\ t \in [0,T], \\ a^2(x,0) = 0, & for\ x \in \mathbb{R}^+, \end{cases} \tag{2.2.40}$$

where

$$\mathcal{G}(a^0) = \bar{B}(x,t)\partial_x^2 a^0 + 2\check{B}(x,t)\partial_x a^0 + \hat{B}(x,t)a^0 - S^0(x,t) - S^1(x,t) - S^2(x,t).$$

Let $\tilde{a}^2 = a^2 + \bar{B}^2 \triangleq a^2 + e^{-x^2}(0, B_1^2, 0)^T(0,t)$. Then $\tilde{a}^2$ solves the following initial boundary value problem

$$\begin{cases} \partial_t \tilde{a}^2 + \mathcal{L}^0 \tilde{a}^2 = \mathcal{P}(a^0, \bar{B}^2), & in\ \mathbb{R}^+ \times [0,T], \\ M^0 \tilde{a}^2 = 0, & for\ t \in [0,T], \\ \tilde{a}^2(x,0) = 0, & for\ x \in \mathbb{R}^+, \end{cases} \tag{2.2.41}$$

where

$$\mathcal{P}(a^0, \bar{B}^2) = \mathcal{G}(a^0) + \partial_t \bar{B}^2 + \mathcal{L}^0 \bar{B}^2.$$

It follows from 1.2.1, 2.2.38 and 2.2.39 that

$$\mathcal{P}(a^0, \bar{B}^2) \in \bigcap_{j=0}^{[\frac{m}{2}]-5} C^j([0,T]; H^{[\frac{m}{2}]-5-j}(\mathbb{R}^+)).$$

and

$$\partial_t^k \mathcal{P}(a^0, \bar{B}^2)(x,0) = 0, \quad k = 0, 1, \cdots, \left[\frac{m}{2}\right] - 5,\ for\ x \in \mathbb{R}^+.$$

Following the argument for a^1, there exists a unique solution a^2 to (2.2.40) such that

$$a^2 \in \bigcap_{j=0}^{[\frac{m}{2}]-5} C^j([0,T]; H^{[\frac{m}{2}]-5-j}(\mathbb{R}^+)). \tag{2.2.42}$$

Finally, we determine B_0^2 by requiring the $\mathcal{O}(\varepsilon^2)$-order term in (2.1.5) to vanish, i.e. B_0^2 is the solution to the following initial boundary value problem for the linear uniform parabolic equation

$$\begin{cases} \partial_t B_0^2 - b_0 \partial_y^2 B_0^2 + (\omega_0 + \tau_0)(0,t) B_0^2 + G(B^0, B^1, (B^2)_{II}) = 0, & in\ \mathbb{R}^+ \times [0,T], \\ B_0^2(0,t) = -a_0^2(0,t), & for\ t \in [0,T], \\ B_0^2(y,0) = 0, & for\ y \in \mathbb{R}^+. \end{cases} \tag{2.2.43}$$

where

$$
\begin{aligned}
G(B^0, B^1, (B^2)_{II}) =& \frac{1}{6}y^3\partial_x^3 u(0,t)\partial_y B_0^0 + \frac{1}{2}y^2\partial_x^2 u(0,t)\partial_y B_0^1 + \frac{1}{2}y^2\partial_x^2\omega_0(0,t)\partial_y B_0^0 \\
&+ y\partial_x\omega_0(0,t)B_0^1 + y\partial_x\omega_1(0,t)B_2^1 + (\omega_1+\tau_1)(0,t)B_2^2 \\
&+ \frac{1}{2}y^2\partial_x^2\omega_0(0,t)B_0^0 + y\partial_x\tau_0(0,t)B_0^1 + y\partial_x\tau_1(0,t)B_2^1 \\
&- 2k_0(0,t)\partial_y B_2^1 + S^0(0,t) + S^1(0,t) + S^2(0,t).
\end{aligned}
$$

Notice that

$$
\mathcal{M}^+(a^2+B^2)(0,t) = 0, \quad for\ t\in[0,T]. \tag{2.2.44}
$$

The solution B_0^2 to (2.2.43) satisfies the properties that

$$
\begin{aligned}
&\langle y\rangle^l\partial_t^k\partial_y^\alpha B_0^2 \in C^0([0,T];L^2(\mathbb{R}^+)), \quad j=1,2, \\
&\qquad for\ k+\alpha \le \left[\frac{m}{2}\right]-5,\ l\in\mathbb{N}_0,
\end{aligned} \tag{2.2.45}
$$

Furthermore,

$$
\partial_t^k B_0^2(y,0) = 0, \quad k=0,1,\cdots,\left[\frac{m}{2}\right]-5,\ j=1,2,\ y\in\mathbb{R}^+. \tag{2.2.46}
$$

Thus the construction of a^i and B^i, $i=1,2.$ are completed.

2.3 Truncation Terms

We can now conclude that the approximate solution $V^{app}(x,t)$ defined in (2.1.1) has at least the smoothness such that

$$
V^{app}(x,t) \in \bigcap_{j=0}^{\left[\frac{m}{2}\right]-5} C^j([0,T];H^{\left[\frac{m}{2}\right]-5-j}(\mathbb{R}^+)), \tag{2.3.1}
$$

and it is the solution of the following initial boundary value problem

$$
\begin{aligned}
&\partial_t V^{app}(x,t) + \mathcal{L}^\varepsilon(x,t)V^{app}(x,t) + \mathcal{F}(x,t) = \\
&\qquad \varepsilon^3\mathcal{E}_{in}(\varepsilon,x,t) + \varepsilon^2\mathcal{E}_{bd}(\varepsilon,\frac{x}{\varepsilon},t), \quad in\ \mathbb{R}^+\times[0,T],
\end{aligned} \tag{2.3.2}
$$

$$\mathcal{M}^+ V^{app}(0,t) = \begin{pmatrix} (R - C_v)\theta_1 \\ 0 \end{pmatrix}, \quad for\ t \in [0,T], \tag{2.3.3}$$

$$V^{app}(x,0) = V_0(x), \qquad for\ x \in \mathbb{R}^+, \tag{2.3.4}$$

where

$$\begin{aligned} \mathcal{E}_{in}(\varepsilon,x,t) =& \Big(- \tilde{\mathcal{B}}(x,t)\partial_x^2 a^1 - 2\breve{\mathcal{B}}(x,t)\partial_x a^1 - \hat{\mathcal{B}}(x,t) a^1 \Big) \\ & + \varepsilon \Big(- \partial_x^2 a^2 - 2\breve{\mathcal{B}}(x,t)\partial_x a^2 - \hat{\mathcal{B}}(x,t) a^2 \Big), \end{aligned} \tag{2.3.5}$$

and

$$\mathcal{E}_{bd}(\varepsilon, \frac{x}{\varepsilon}, t) = \left[\sum_{i=4}^{7} K^i \right]_{II} (t), \tag{2.3.6}$$

here

$$[K]_{II}(t) = (0, K_1, K_2)^T. \tag{2.3.7}$$

For any $\varepsilon > 0$,

$$\mathcal{E}_{in}(\varepsilon,x,t) \in \bigcap_{j=0}^{[\frac{m}{2}]-7} C^j([0,T]; H^{[\frac{m}{2}]-7-j}(\mathbb{R}^+)), \tag{2.3.8}$$

and

$$\partial_t^k \mathcal{E}_{in}(\varepsilon,x,0) = 0, \quad k = 0,1,\ x \in \mathbb{R}^+. \tag{2.3.9}$$

and

$$\mathcal{E}_{bd}(\varepsilon, \frac{x}{\varepsilon}, t) \in \bigcap_{j=0}^{[\frac{m}{2}]-7} C^j([0,T]; H^{[\frac{m}{2}]-7-j}(\mathbb{R}^+)), \tag{2.3.10}$$

$$\partial_t^k \mathcal{E}_{bd}(\varepsilon, \frac{x}{\varepsilon}, t) = 0, \quad k = 0,1,\cdots,\left[\frac{m}{2}\right] - 6, \quad for\ x \in \mathbb{R}^+. \tag{2.3.11}$$

Here (2.3.10) follows from the structure of $\mathcal{E}_{bd}$. Moreover, we have

$$\sup_{t\in[0,T]} \sum_{k+\alpha \le [\frac{m}{2}]-7} \|\partial_t^k \partial_x^\alpha \mathcal{E}_{in}(\varepsilon,x,t)\|^2_{L^2(\mathbb{R}^+)} \le C, \tag{2.3.12}$$

$$\sup_{t\in[0,T]} \sum_{k+\alpha \le [\frac{m}{2}]-6} \|\langle y\rangle^l \partial_t^k \partial_x^\alpha \mathcal{E}_{bd}(\varepsilon,x,t)\|^2_{L^2(\mathbb{R}^+)} \le C. \tag{2.3.13}$$

In the following, we denote for simplicity $\|\cdot\|$ the norm for $L^2(\mathbb{R}^+)$.

Chapter 3

Estimates of the Error Term of the Approximate Solution and Main Results

In this chapter, we first derive the initial boundary problem of the error term, then by an energy estimate, we show the pointwise estimate of the error term of the approximate solution, which readily yields the uniform stability result for the linearized Navier-Stokes solution in the zero-dissipation limit.

3.1 Error Equations

The procedure of the error estimates is carried out by the following steps. Firstly, we give the basic L^2 estimate of the error term itself (see Lemma 3.2.1). Secondly, we then give the estimate of the tangential derivatives of the error term (see Lemma 1.2.1). Finally, using the equations (3.1.16)–(3.1.17) and the previous estimates, we obtain the L^2 estimate of the normal derivatives of the error term.(see Lemma 3.2.5).

Define the error term $\varphi^\varepsilon = (\varphi_0^\varepsilon, \varphi_1^\varepsilon, \varphi_2^\varepsilon)^T$ of the approximate solution for the linearized Navier-Stokes equations (1.2.2)–(1.2.4). That is,

$$\varphi^\varepsilon(x,t) = V^\varepsilon(x,t) - V^{app}(x,t), \quad for\ (x,t) \in \mathbb{R}^+ \times [0,T]. \tag{3.1.1}$$

whose existence is guaranteed by the discussion in chapter 1, and together with (2.3.1) and Proposition 1.2.1, and we have

$$\varphi^\varepsilon(x,t) \in \bigcap_{j=0}^{3} C^j([0,T]; H^{3-j}(\mathbb{R}^+)). \tag{3.1.2}$$

Furthermore, from (1.2.2)–(1.2.4) and (2.3.2)–(2.3.4), it can be seen that φ^ε solves the following initial boundary value problem

$$\begin{cases} \partial_t \varphi^\varepsilon(x,t) + \mathcal{L}^\varepsilon \varphi^\varepsilon(x,t) = \varepsilon^2 J(\varepsilon, \frac{x}{\varepsilon}, t), & in\ \mathbb{R}^+ \times [0,T], \\ \mathcal{M}^+ \varphi^\varepsilon(0,t) = 0, & for\ t \in [0,T], \\ \varphi^\varepsilon(x,0) = 0, & for\ x \in \mathbb{R}^+, \end{cases} \tag{3.1.3}$$

where

$$\begin{aligned} J(\varepsilon, \frac{x}{\varepsilon}, t) &= (J_0, J_1, J_2)^T(\varepsilon, \frac{x}{\varepsilon}, t) \\ &= \varepsilon \mathcal{E}_{in}(\varepsilon, x, t) + \mathcal{E}_{bd}(\varepsilon, \frac{x}{\varepsilon}, t) \end{aligned} \tag{3.1.4}$$

and $\mathcal{L}^\varepsilon$ is the same definition as in (1.2.5). By (2.3.8)–(2.3.11) and the assumption $m \geq 13$, we have

$$J(\varepsilon, \frac{x}{\varepsilon}, t) \in \bigcap_{j=0}^{2} C^j([0,T]; H^{2-j}(\mathbb{R}^+)), \tag{3.1.5}$$

$$\partial_t^k J(\varepsilon, \frac{x}{\varepsilon}, 0) = 0, \quad k = 0, 1,\ x \in \mathbb{R}^+. \tag{3.1.6}$$

Then it follows from (2.3.12), (3.1.5) and (3.1.6) that for $0 \leq t \leq T$,

$$\|J(\varepsilon,t)\|^2 \leq C\varepsilon, \tag{3.1.7}$$

$$\|\partial_x J(\varepsilon,t)\|^2 \leq C\varepsilon, \tag{3.1.8}$$

$$\|\partial_t J(\varepsilon,t)\|^2 \leq C\varepsilon, \tag{3.1.9}$$

and

$$\|x\partial_x J(\varepsilon,t)\|^2 \le C\varepsilon. \tag{3.1.10}$$

Hereafter, we write $\varphi = (\varphi_0, \varphi_1, \varphi_2)^T$ instead of φ^ε for simplicity. The equations that φ satisfies are

$$\begin{aligned}
&\begin{pmatrix} \partial_t\varphi_0 \\ \partial_t\varphi_1 \\ \partial_t\varphi_2 \end{pmatrix} + \begin{pmatrix} u & 0 & 0 \\ 0 & u & \frac{\theta}{\varrho} \\ 0 & \frac{c^2\varrho}{\theta} & u \end{pmatrix} \begin{pmatrix} \partial_x\varphi_0 \\ \partial_x\varphi_1 \\ \partial_x\varphi_2 \end{pmatrix} \\
&+ \begin{pmatrix} \omega_0 & 0 & \omega_1 \\ 0 & 0 & 0 \\ \omega_2 & 0 & \omega_3 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{pmatrix} + \begin{pmatrix} \tau_0 & 0 & \tau_1 \\ \tau_2 & 0 & \tau_3 \\ \tau_4 & 0 & \tau_5 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{pmatrix} \\
=&\varepsilon^2 \begin{pmatrix} b_0 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_x^2\varphi_0 \\ \partial_x^2\varphi_1 \\ \partial_x^2\varphi_2 \end{pmatrix} + 2\varepsilon^2 \begin{pmatrix} 0 & 0 & k_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_x\varphi_0 \\ \partial_x\varphi_1 \\ \partial_x\varphi_2 \end{pmatrix} \\
&+ \varepsilon^2 \begin{pmatrix} 0 & 0 & k_2 \\ 0 & 0 & 0 \\ 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} J_0 \\ J_1 \\ J_2 \end{pmatrix}
\end{aligned} \tag{3.1.11}$$

and

$$\varphi_0(0,t) = \varphi_1(0,t) = 0, \qquad for\ t \in t \in [0,T], \tag{3.1.12}$$

$$(\varphi_0, \varphi_1, \varphi_2)^T(x,0) = 0, \qquad for\ x \in \mathbb{R}^+. \tag{3.1.13}$$

Multiplying the matrix $\mathcal{A}_0(x,t)$ defined in (2.2.4) to the equations (3.1.11), we get the following symmetric system

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} \partial_t\varphi_0 \\ \partial_t\varphi_1 \\ \partial_t\varphi_2 \end{pmatrix} + \begin{pmatrix} u & 0 & 0 \\ 0 & \beta u & 1 \\ 0 & 1 & \delta u \end{pmatrix} \begin{pmatrix} \partial_x\varphi_0 \\ \partial_x\varphi_1 \\ \partial_x\varphi_2 \end{pmatrix} \\
&+ \begin{pmatrix} \omega_0 & 0 & \omega_1 \\ 0 & 0 & 0 \\ \delta\omega_2 & 0 & \delta\omega_3 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{pmatrix} + \begin{pmatrix} \tau_0 & 0 & \tau_1 \\ \beta\tau_2 & 0 & \beta\tau_3 \\ \delta\tau_4 & 0 & \delta\tau_5 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{pmatrix}
\end{aligned}$$

$$
=\varepsilon^2 \begin{pmatrix} b_0 & 0 & 0 \\ 0 & \beta b_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_x^2\varphi_0 \\ \partial_x^2\varphi_1 \\ \partial_x^2\varphi_2 \end{pmatrix} + 2\varepsilon^2 \begin{pmatrix} 0 & 0 & k_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_x\varphi_0 \\ \partial_x\varphi_1 \\ \partial_x\varphi_2 \end{pmatrix}
$$

$$
+\varepsilon^2 \begin{pmatrix} 0 & 0 & k_2 \\ 0 & 0 & 0 \\ 0 & 0 & \delta k_3 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} J_0 \\ \beta J_1 \\ \delta J_2 \end{pmatrix}, \tag{3.1.14}
$$

which can be written as

$$
\begin{aligned}
&\partial_t\varphi_0 + u\partial_x\varphi_0 + (\omega_0+\tau_0)\varphi_0 + (\omega_1+\tau_1)\varphi_2 \\
&\qquad = \varepsilon^2 b_0\partial_x^2\varphi_0 + 2\varepsilon^2 k_0\partial_x\varphi_2 + \varepsilon^2 k_2\varphi_2 + \varepsilon^2 J_0,
\end{aligned} \tag{3.1.15}
$$

$$
\beta\partial_t\varphi_1 + \delta\partial_x\varphi_1 + \partial_x\varphi_2 + \beta\tau_2\varphi_0 + \beta\tau_3\varphi_2 = \varepsilon^2\beta b_2\partial_x^2\varphi_1 + \varepsilon^2\beta J_1, \tag{3.1.16}
$$

$$
\begin{aligned}
&\delta\partial_t\varphi_2 + \partial_x\varphi_1 + \delta u\partial_x\varphi_2 + \delta(\omega_2+\tau_4)\varphi_0 + \delta(\omega_3+\tau_5)\varphi_2 \\
&\qquad = \varepsilon^2\delta k_3\varphi_2 + \varepsilon^2\delta J_2.
\end{aligned} \tag{3.1.17}
$$

where β and δ are the same as (2.2.4).

3.2 Energy Estimates

In this section, we carry out the energy estimate step by step. In the following, we use the notation that

$$
\|\varphi(t)\|^2_{A_0(t)} = \big(\varphi_0(t),\varphi_0(t)\big) + \big(\beta\varphi_1(t),\varphi_1(t)\big) + \big(\delta\varphi_2(t),\varphi_2(t)\big),
$$

with $(\cdot,\cdot)$ denotes the inner product in $L^2(\mathbb{R}^+)$.

3.2.1 Basic L^2 Estimates

Lemma 3.2.1 *The error term φ satisfies*

$$
\sup_{0\le t\le T}\|\varphi(t)\|^2 + C\varepsilon^2\sum_{j=0}^{1}\int_0^T \|\partial_x\varphi_j(\tau)\|^2 d\tau \le C\varepsilon^5, \tag{3.2.1}
$$

where C is a uniform constant depending on $\sup_{\mathbb{R}^+\times[0,T]}(|V'|, |\partial_x V'|, |\partial_t V'|)$.

Proof: Taking the inner product in $L^2(\mathbb{R}^+)$ of (3.1.15) with φ_0, we get by integration by parts that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\varphi_0(t)\|^2 + \varepsilon^2 b_0 \int_{\mathbb{R}^+} |\partial_x\varphi_0|^2 dx \\
=&\frac{1}{2}\int_{\mathbb{R}^+} \partial_x u\varphi_0^2 dx - \int_{\mathbb{R}^+}(\omega_0+\tau_0)\varphi_0^2 dx - \int_{\mathbb{R}^+}(\omega_1+\tau_1)\varphi_2\varphi_0 dx \\
&+ 2\varepsilon^2\int_{\mathbb{R}^+} k_0\partial_x\varphi_2\varphi_0 dx + \varepsilon^2\int_{\mathbb{R}^+} k_2\varphi_2\varphi_0 dx + \varepsilon^2\int_{\mathbb{R}^+} J_0\varphi_0 dx,
\end{aligned}$$

Using Young's inequality yields

$$\begin{aligned}
&\frac{d}{dt}\|\varphi_0\|^2 + C\varepsilon^2\int_{\mathbb{R}^+} |\partial_x\varphi_0|^2 dx \\
\le& C\int_{\mathbb{R}^+} |\varphi_0|^2 dx + C\int_{\mathbb{R}^+} |\varphi_2|^2 dx + C\varepsilon^4\int_{\mathbb{R}^+} |J_0|^2 dx,
\end{aligned} \tag{3.2.2}$$

where the boundary condition (3.1.12) have been used. Multiplying φ_1 to (3.1.16), and integrating by parts with the condition (3.1.12) give that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^+}\beta\varphi_1^2 dx - \frac{1}{2}\int_{\mathbb{R}^+}\partial_t\beta\varphi_1^2 dx - \frac{1}{2}\int_{\mathbb{R}^+}\partial_x(\beta u)\varphi_1^2 dx + \int_{\mathbb{R}^+}\partial_x\varphi_2\cdot\varphi_1 dx \\
&+\int_{\mathbb{R}^+}\beta\tau_2\varphi_0\varphi_1 dx + \int_{\mathbb{R}^+}\beta\tau_3\varphi_2\varphi_1 dx + \varepsilon^2\beta b_2\int_{\mathbb{R}^+}|\partial_x\varphi_1|^2 dx = \varepsilon^2\int_{\mathbb{R}^+}\beta J_1\varphi_1 dx.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^+}\beta\varphi_1^2 dx + \beta b_2\varepsilon^2\int_{\mathbb{R}^+}|\partial_x\varphi_1|^2 dx + \int_{\mathbb{R}^+}\partial_x\varphi_2\cdot\varphi_1 dx \\
\le& C\int_{\mathbb{R}^+}|\varphi_0|^2 dx + C\int_{\mathbb{R}^+}|\varphi_1|^2 dx + C\int_{\mathbb{R}^+}|\varphi_2|^2 dx + C\varepsilon^4\int_{\mathbb{R}^+}|J_1|^2 dx,
\end{aligned} \tag{3.2.3}$$

where C is a constant depending on $\sup_{\mathbb{R}^+\times[0,T]}(|V'|, |\partial_x V'|, |\partial_t V'|)$. Similarly, due to assumption $u(0,t) = 0$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^+}\delta\varphi_2^2 dx - \int_{\mathbb{R}^+}\partial_x\varphi_2\cdot\varphi_1 dx \\
\le& C\int_{\mathbb{R}^+}|\varphi_0|^2 dx + C\int_{\mathbb{R}^+}|\varphi_2|^2 dx + C\varepsilon^4\int_{\mathbb{R}^+}|J_2|^2 dx.
\end{aligned} \tag{3.2.4}$$

Summing (3.2.2)–(3.2.4) up gives

$$\frac{1}{2}\frac{d}{dt}\|\varphi(t)\|^2_{\mathcal{A}_0(t)} + C\varepsilon^2\sum_{j=0}^{1}\|\partial_x\varphi_j(t)\|^2 \le C\|\varphi(t)\|^2 + \varepsilon^4\|J(\varepsilon,t)\|^2. \tag{3.2.5}$$

By virtue of (1.2.1) and the fact that β and δ are positive and continuous, then there exists a constant C_0 depending on $\sup_{\mathbb{R}^+\times[0,T]}|V'|$ such that

$$2C_0^{-1}\|\varphi(t)\|^2 \le \|\varphi(t)\|^2_{\mathcal{A}_0(t)} \le C_0\|\varphi(t)\|^2, \quad for\ 0\le t\le T. \tag{3.2.6}$$

Hence,

$$\begin{aligned}&\|\varphi(t)\|^2 + C\varepsilon^2\sum_{j=0}^{1}\int_0^t \|\partial_x\varphi_j(\tau)\|^2 d\tau\\ \le& C\int_0^t \|\varphi(\tau)\|^2 + C\varepsilon^4\int_0^t \|J(\varepsilon,\tau)\|^2 d\tau, \quad for\ 0\le t\le T.\end{aligned} \tag{3.2.7}$$

In view of (2.3.13), we have

$$\int_{\mathbb{R}^+}|\mathcal{E}_{bd}(\varepsilon,\frac{x}{\varepsilon},t)|dx \le C\varepsilon, \quad for\ 0\le t\le T,$$

which together with (3.1.7) and utilizing Gronwall's inequality and then taking the supreme of t, we conclude that

$$\sup_{0\le t\le T}\|\varphi(t)\|^2 + C\varepsilon^2\sum_{j=0}^{1}\int_0^T \|\partial_x\varphi_j(\tau)\|^2 d\tau \le C\varepsilon^5.$$

□

3.2.2 Tangential Derivatives Estimates

We call $(\partial_t\varphi, x\partial_x\varphi)$ the tangential derivatives of φ, and denote it by $D_{tan}\varphi$. Before giving the tangential derivatives of φ, we first list the following two lemmas.

Lemma 3.2.2 *For some uniform constant* C *depending on* $\sup_{\mathbb{R}^+\times[0,T]}(|V'|,|\partial_x V'|,|\partial_t V'|,|\partial_{xt}^2 V'|)$, *we have*

$$\begin{aligned}&\frac{d}{dt}\|\psi(t)\|^2 + C\varepsilon^2\sum_{j=0}^{1}\int_{\mathbb{R}^+}|\partial_x\psi_j(t)|^2 dx\\ \le& C\|\psi(t)\|^2 + C\|\varphi(t)\|^2 + C\|x\partial_x\varphi(t)\|^2 + C\varepsilon^3,\end{aligned} \tag{3.2.8}$$

where $\psi(x,t) = \partial_t\varphi(x,t) \triangleq (\psi_0,\psi_1,\psi_2)^T$.

Proof: Using the notation $\psi = \partial_t\varphi = (\psi_0, \psi_1, \psi_2)^T$, we derive from the equations (3.1.15)–(3.1.17) and (2.3.11), ψ satisfies the following initial and boundary conditions

$$\psi_0(0,t) = \psi_1(0,t) = 0, \tag{3.2.9}$$

and

$$\lim_{t\to 0^+} \psi(x,t) = 0. \tag{3.2.10}$$

It follows from (3.1.15) that ψ_0 solves

$$\begin{aligned}
&\partial_t\psi_0 - \varepsilon^2 b_0\partial_x^2\psi_0 + u\partial_x\psi_0 + (\omega_0+\tau_0)\psi_0 + (\omega_1+\tau_1)\psi_2 \\
=&2\varepsilon^2\partial_t k_0\partial_x\varphi_2 + \varepsilon^2\partial_t k_2\varphi_2 + 2\varepsilon^2 k_0\partial_x\psi_2 - \partial_t(\omega_1+\tau_1)\varphi_2 \\
&- \partial_t u\partial_x\varphi_0 - \partial_t(\omega_0+\tau_0)\varphi_0 + \varepsilon^2 k_2\psi_2 + \varepsilon^2\partial_t J_0.
\end{aligned} \tag{3.2.11}$$

We multiply the above equation by ψ_0 and integrate the resulting equation over $\mathbb{R}^+$ to obtain after integration by parts that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\psi_0\|^2 + \varepsilon^2 b_0\int_{\mathbb{R}^+}|\partial_x\psi_0|^2dx - \frac{1}{2}\int_{\mathbb{R}^+}\partial_x u|\psi_0|^2dx + \int_{\mathbb{R}^+}(\omega_0+\tau_0)|\psi_0|^2dx \\
=&2\varepsilon^2\int_{\mathbb{R}^+}\partial_t k_0\partial_x\varphi_2\cdot\psi_0dx + \varepsilon^2\int_{\mathbb{R}^+}\partial_t k_2\varphi_2\psi_0dx + 2\varepsilon^2\int_{\mathbb{R}^+}k_0\partial_x\psi_2\cdot\psi_0dx \\
&- \int_{\mathbb{R}^+}\partial_t(\omega_1+\tau_1)\varphi_2\psi_0dx - \int_{\mathbb{R}^+}\partial_t u\partial_x\varphi_0\cdot\psi_0dx - \int_{\mathbb{R}^+}\partial_t(\omega_0+\tau_0)\varphi_0\psi_0dx \\
&+ \varepsilon^2\int_{\mathbb{R}^+}k_2\psi_2\psi_0dx + \varepsilon^2\int_{\mathbb{R}^+}\partial_t J_0\psi_0dx - \int_{\mathbb{R}^+}(\omega_1+\tau_1)\psi_2\psi_0dx \\
\le&C\varepsilon^2\int_{\mathbb{R}^+}\partial_x\varphi_2\cdot\psi_0 + C\int_{\mathbb{R}^+}\varphi_2\psi_0dx + C\varepsilon^2\int_{\mathbb{R}^+}\partial_x\psi_2\cdot\psi_0dx \\
&+ C\int_{\mathbb{R}^+}\partial_x\varphi_0\cdot\psi_0dx + C\int_{\mathbb{R}^+}\varphi_0\psi_0dx + C\int_{\mathbb{R}^+}\psi_2\psi_0dx + \varepsilon^2\int_{\mathbb{R}^+}\partial_t J_0\cdot\psi_0dx \\
\le&\frac{b_0\varepsilon^2}{2}\int_{\mathbb{R}^+}|\partial_x\psi_0|^2dx + C\int_{\mathbb{R}^+}|\varphi_2|^2dx + C\int_{\mathbb{R}^+}|\psi_0|^2dx \\
&+ C\int_{\mathbb{R}^+}|\psi_2|^2dx + C\int_{\mathbb{R}^+}|\partial_x\varphi_0|^2dx + C\int_{\mathbb{R}^+}|\varphi_0|^2dx + C\varepsilon^4\int_{\mathbb{R}^+}|\partial_t J_0|^2dx.
\end{aligned} \tag{3.2.12}$$

where we have used the Young's inequality. Then

$$\frac{1}{2}\frac{d}{dt}\|\psi_0\|^2 + C\varepsilon^2\int_{\mathbb{R}^+}|\partial_x\psi_0|^2dx$$

$$
\begin{aligned}
\leq & C\int_{\mathbb{R}^+}|\varphi_0|^2dx+C\int_{\mathbb{R}^+}|\varphi_2|^2dx+C\int_{\mathbb{R}^+}|\psi_0|^2dx\\
&+C\int_{\mathbb{R}^+}|\psi_2|^2dx+C\int_{\mathbb{R}^+}|\partial_x\varphi_0|^2dx+C\varepsilon^4\int_{\mathbb{R}^+}|\partial_tJ_0|^2dx. \qquad (3.2.13)
\end{aligned}
$$

Similarly, it follows from the equation (3.1.16) that

$$
\begin{aligned}
&\beta\partial_t\psi_1+\beta u\partial_x\psi_1-\varepsilon^2\beta b_2\partial_x^2\psi_1+\partial_t\beta\psi_1\\
=&\varepsilon^2\partial_t(\beta b_2)\partial_x^2\varphi_1+\varepsilon^2\partial_t\beta J_1+\varepsilon^2\beta\partial_tJ_1-\partial_t(\beta u)\partial_x\varphi_1\\
&-\partial_x\psi_2-\partial_t(\beta\tau_2)\varphi_0-\partial_t(\beta\tau_3)\varphi_2-\beta\tau_2\psi_0-\beta\tau_3\psi_2. \qquad (3.2.14)
\end{aligned}
$$

Taking inner product with ψ_1 in $L^2(\mathbb{R}^+)$ and integrating by parts yield

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^+}\beta|\psi_1|^2dx-\frac{1}{2}\int_{\mathbb{R}^+}\partial_t\beta|\psi_1|^2dx+\int_{\mathbb{R}^+}\partial_x\psi_2\cdot\psi_1dx\\
&-\frac{1}{2}\int_{\mathbb{R}^+}\partial_x(\beta u)|\psi_1|^2dx+C_1\varepsilon^2\int_{\mathbb{R}^+}|\partial_x\psi_1|^2dx+\int_{\mathbb{R}^+}\partial_t\beta|\psi_1|^2dx\\
\leq&\varepsilon^2\int_{\mathbb{R}^+}\partial_t(\beta b_2)\partial_x^2\varphi_1\psi_1dx+\varepsilon^2\int_{\mathbb{R}^+}\partial_t\beta J_1\psi_1dx\\
&+\varepsilon^2\int_{\mathbb{R}^+}\beta\partial_tJ_1\cdot\psi_1dx-\int_{\mathbb{R}^+}\partial_t(\beta u)\partial_x\varphi_1\cdot\psi_1dx-\int_{\mathbb{R}^+}\beta\tau_2\psi_0\psi_1dx\\
&-\int_{\mathbb{R}^+}\partial_t(\beta\tau_2)\varphi_2\psi_1dx-\int_{\mathbb{R}^+}\partial_t(\beta\tau_3)\varphi_2\psi_1dx-\int_{\mathbb{R}^+}\beta\tau_3|\psi_2|^2dx\\
\leq&C\varepsilon^2\int_{\mathbb{R}^+}\partial_x\varphi_1\partial_x\psi_1dx+C\varepsilon^2\int_{\mathbb{R}^+}J_1\psi_1dx+C\varepsilon^2\int_{\mathbb{R}^+}\partial_tJ_1\psi_1dx+C\int_{\mathbb{R}^+}\partial_x\varphi_1\cdot\psi_1dx\\
&+C\int_{\mathbb{R}^+}\varphi_2\psi_1dx+C\int_{\mathbb{R}^+}\psi_0\psi_1dx+C\int_{\mathbb{R}^+}|\psi_2|^2dx\\
\leq&\frac{C_1\varepsilon^2}{2}\int_{\mathbb{R}^+}|\partial_x\psi_1|^2dx+C\int_{\mathbb{R}^+}|\partial_x\varphi_1|^2dx+C\varepsilon^4\int_{\mathbb{R}^+}|J_1|^2dx+C\varepsilon^4\int_{\mathbb{R}^+}|\partial_tJ_1|^2dx\\
&+C\int_{\mathbb{R}^+}|\psi_1|^2dx+C\int_{\mathbb{R}^+}|\psi_0|^2dx+C\int_{\mathbb{R}^+}|\varphi_2|^2dx+C\int_{\mathbb{R}^+}|\psi_2|^2dx, \qquad (3.2.15)
\end{aligned}
$$

where C_1 is a uniform constant which is independent of ε.

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^+}\beta|\psi_1|^2dx+C\varepsilon^2\int_{\mathbb{R}^+}|\partial_x\psi_1|^2dx+\int_{\mathbb{R}^+}\partial_x\psi_2\psi_1dx\\
\leq&C\int_{\mathbb{R}^+}|\varphi_2|^2dx+C\int_{\mathbb{R}^+}|\psi_0|^2dx+C\int_{\mathbb{R}^+}|\psi_1|^2dx+C\int_{\mathbb{R}^+}|\psi_2|^2dx\\
&+C\varepsilon^4\int_{\mathbb{R}^+}|J_1|^2dx+C\varepsilon^4\int_{\mathbb{R}^+}|\partial_tJ_1|^2dx++C\int_{\mathbb{R}^+}|\partial_x\varphi_1|^2dx. \qquad (3.2.16)
\end{aligned}
$$

It is easy to check that ψ_2 satisfies

$$\begin{aligned}
&\delta\partial_t\psi_2+\delta u\partial_x\psi_2+(\delta\omega_3+\delta\tau_5)\psi_2+\partial_t\delta\psi_2-\varepsilon^2\delta k_3\psi_2\\
=&\varepsilon^2\partial_t(\delta k_3)\varphi_2+\varepsilon^2\partial_t\delta J_2+\varepsilon^2\delta\partial_t J_2-\partial_x\psi_1-\partial_t(\delta u)\partial_x\varphi_2\\
&-\partial_t(\delta\omega_2+\delta\tau_4)\varphi_0-\partial_t(\delta\omega_3+\delta\tau_5)\varphi_2+(\delta\omega_2+\delta\tau_4)\psi_0.
\end{aligned}\tag{3.2.17}$$

We multiply ψ_2 on both sides and integrate over $\mathbb{R}^+$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^+}\delta|\psi_2|^2dx-\frac{1}{2}\int_{\mathbb{R}^+}\partial_t\delta|\psi_2|^2dx-\frac{1}{2}\int_{\mathbb{R}^+}\partial_x(\delta u)|\psi_2|^2dx\\
&+\int_{\mathbb{R}^+}(\delta\omega_3+\delta\tau_5)|\psi_2|^2dx+\int_{\mathbb{R}^+}\partial_t\delta|\psi_2|^2dx-\varepsilon^2\int_{\mathbb{R}^+}\delta k_3|\psi_2|^2dx\\
=&\varepsilon^2\int_{\mathbb{R}^+}\partial_t(\delta k_3)\varphi_2\psi_2dx+\varepsilon^2\int_{\mathbb{R}^+}\partial_t\delta J_2\psi_2dx+\varepsilon^2\int_{\mathbb{R}^+}\delta\partial_t J_2\psi_2dx\\
&-\int_{\mathbb{R}^+}\partial_t(\delta u)\partial_x\varphi_2\cdot\psi_2dx-\int_{\mathbb{R}^+}\partial_t(\delta\omega_2+\delta\tau_4)\varphi_0\cdot\psi_2dx\\
&-\int_{\mathbb{R}^+}\partial_x\psi_1\cdot\psi_2dx-\int_{\mathbb{R}^+}\partial_t(\delta\omega_3+\delta\tau_5)\varphi_2\psi_2dx+\int_{\mathbb{R}^+}(\delta\omega_2+\delta\tau_4)\psi_0\psi_2dx.
\end{aligned}\tag{3.2.18}$$

That is

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^+}\delta|\psi_2|^2dx+\int_{\mathbb{R}^+}\partial_x\psi_1\cdot\psi_2dx\\
\le& I_0+C\int_{\mathbb{R}^+}|\varphi_0|^2dx+C\int_{\mathbb{R}^+}|\varphi_2|^2dx+C\int_{\mathbb{R}^+}|\psi_0|^2dx\\
&+C\varepsilon^4\int_{\mathbb{R}^+}|J_2|^2dx+C\varepsilon^4\int_{\mathbb{R}^+}|\partial_tJ_2|^2dx+C\int_{\mathbb{R}^+}|\psi_2|^2dx,
\end{aligned}\tag{3.2.19}$$

where

$$I_0=\int_{\mathbb{R}^+}\partial_t(\delta u)\partial_x\varphi_2\cdot\psi_2dx.$$

By virtue of $u(0,t)=0$ and $\partial_t u(0,t)=0$, one finds that for $\xi\in(0,x)$,

$$\begin{aligned}
I_0=&\int_{\mathbb{R}^+}\partial_t(\delta u)\partial_x\varphi_2\cdot\psi_2dx-\int_{\mathbb{R}^+}\big(\partial_t(\delta u)(0,t)\big)\partial_x\varphi_2\cdot\psi_2dx\\
=&\int_{\mathbb{R}^+}x\partial_{xt}^2\big(\delta u(\xi,t)\big)\partial_x\varphi_2\cdot\psi_2dx\le C\int_{\mathbb{R}^+}|x\partial_x\varphi_2|^2dx+C\int_{\mathbb{R}^+}|\psi_2|^2dx.
\end{aligned}$$

Then we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^+}\delta|\psi_2|^2dx+\int_{\mathbb{R}^+}\partial_x\psi_1\cdot\psi_2dx$$

$$
\begin{aligned}
\leq & C\int_{\mathbb{R}^+}|\varphi_0|^2dx + C\int_{\mathbb{R}^+}|\varphi_2|^2dx + C\int_{\mathbb{R}^+}|\psi_0|^2dx + C\int_{\mathbb{R}^+}|\psi_2|^2dx \\
& + C\varepsilon^4\int_{\mathbb{R}^+}|J_2|^2dx + C\varepsilon^4\int_{\mathbb{R}^+}|\partial_t J_2|^2dx + C\int_{\mathbb{R}^+}|x\partial_x\varphi_2|^2dx. \qquad (3.2.20)
\end{aligned}
$$

Summing (3.2.13), (3.2.16) and (3.2.20) up, and noticing that

$$
\int_{\mathbb{R}^+}\partial_x\psi_2\cdot\psi_1dx = -\int_{\mathbb{R}^+}\partial_x\psi_1\cdot\psi_2dx,
$$

due to the boundary condition (3.2.9), we get

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|\psi(t)\|^2_{\mathcal{A}_0(t)} + C\varepsilon^2\sum_{j=0}^{1}\|\partial_x\psi_j(t)\|^2 \\
\leq & C\|\psi(t)\|^2 + C\|\varphi(t)\|^2 + C\varepsilon^4\|J(\varepsilon,t)\|^2 + C\varepsilon^4\|\partial_t J(\varepsilon,t)\|^2 \\
& + C\sum_{j=0}^{1}\|\partial_x\varphi_j(t)\|^2 + C\|x\partial_x\varphi(t)\|^2. \qquad (3.2.21)
\end{aligned}
$$

Then by fact 3.1.7, 3.1.9 and together with Lemma 3.2.1, we conclude

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|\psi(t)\|^2 + C\varepsilon^2\sum_{j=0}^{1}\|\partial_x\psi_j(t)\|^2 \\
\leq & C\|\psi(t)\|^2 + C\varepsilon^4\|J(\varepsilon,t)\|^2 + C\varepsilon^4\|\partial_t J(\varepsilon,t)\|^2 \\
& + C\|\varphi(t)\|^2 + C\sum_{j=0}^{1}\|\partial_x\varphi_j(t)\|^2dx + C\|x\partial_x\varphi(t)\|^2 \\
\leq & C\|\psi(t)\|^2 + C\|x\partial_x\varphi(t)\|^2 + C\varepsilon^3. \qquad (3.2.22)
\end{aligned}
$$

Integrating over $[0,T]$, we get the conclusion of the Lemma. □

It requires us to estimate the term $x\partial_x\varphi(t)$, and this is given by the following lemma.

Lemma 3.2.3 *There holds the estimate*

$$
\frac{d}{dt}\|\hat{\phi}(t)\|^2 + C\varepsilon^2\sum_{j=0}^{1}\|\partial_x\hat{\phi}_j(t)\|^2 \leq C\|\hat{\phi}(t)\|^2 + C\|\partial_t\varphi(t)\|^2 + C\varepsilon^3, \qquad (3.2.23)
$$

where $\hat{\phi}(x,t) = x\partial_x\varphi \triangleq (\hat{\phi}_0,\hat{\phi}_1,\hat{\phi}_2)^T$*, and* C *is a uniform constant depending on the bounds of* $V', \partial_x V', \partial_t V', \partial^2_{xt}V', c_0, c_1$ *and* $x\partial_x V'$.

Proof: Applying $x\partial_x$ to the equations (3.1.15)-(3.1.17), using the notation $\hat{\phi}(x,t) = x\partial_x\varphi$, one finds that $\hat{\phi}_0$ satisfies the equation

$$\begin{aligned}
&\partial_t\hat{\phi}_0 + \partial_x u\hat{\phi}_0 + xu\partial_x^2\varphi_0 + x\partial_x(\omega_0+\tau_0)\varphi_0 \\
&+ (\omega_0+\tau_0)\hat{\phi}_0 + x\partial_x(\omega_1+\tau_1)\varphi_2 + (\omega_1+\tau_1)\hat{\phi}_2 \\
=&\varepsilon^2 b_0 x\partial_x(\partial_x^2\varphi_0) + 2\varepsilon^2\partial_x k_0\hat{\phi}_2 + 2k_0 x\varepsilon^2\partial_x^2\varphi_2 + \varepsilon^2 k_2\hat{\phi}_2 + \varepsilon^2 x\partial_x k_2\varphi_2 + \varepsilon^2 x\partial_x J_0.
\end{aligned} \tag{3.2.24}$$

Multiplying $\hat{\phi}_0$ to the equation and integrating by parts yield

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\hat{\phi}_0\|^2 =& -\int_{\mathbb{R}^+}\partial_x u|\hat{\phi}_0|^2dx - \int_{\mathbb{R}^+} xu\partial_x^2\varphi_0\cdot\partial_x^2\varphi_0 dx \\
&- \int_{\mathbb{R}^+} x\partial_x(\omega_0+\tau_0)\varphi_0\hat{\phi}_0 dx - \int_{\mathbb{R}^+}(\omega_0+\tau_0)|\hat{\phi}_0|^2dx \\
&- \int_{\mathbb{R}^+} x\partial_x(\omega_1+\tau_1)\varphi_2\hat{\phi}_0 dx - \int_{\mathbb{R}^+}(\omega_1+\tau_1)\hat{\phi}_2\hat{\phi}_0 dx \\
&+ \varepsilon^2 b_0\int_{\mathbb{R}^+} x\partial_x(\partial_x^2\varphi_0)\hat{\phi}_0 dx + 2\varepsilon^2\int_{\mathbb{R}^+}\partial_x k_0\hat{\phi}_2\hat{\phi}_0 dx \\
&+ 2\varepsilon^2\int_{\mathbb{R}^+} k_0 x\partial_x^2\varphi_2\cdot\hat{\phi}_0 dx + \varepsilon^2\int_{\mathbb{R}^+} k_0\hat{\phi}_2\hat{\phi}_0 dx \\
&+ \varepsilon^2\int_{\mathbb{R}^+} x\partial_x k_0\varphi_2\hat{\phi}_0 dx + \varepsilon^2\int_{\mathbb{R}^+} x\partial_x J_0\hat{\phi}_0 dx \\
=&2\varepsilon^2\int_{\mathbb{R}^+}\partial_x k_0\hat{\phi}_2\hat{\phi}_0 dx - \int_{\mathbb{R}^+}\partial_x u|\hat{\phi}_0|^2dx - \int_{\mathbb{R}^+} x\partial_x(\omega_0+\tau_0)\varphi_0\hat{\phi}_0 dx \\
&- \int_{\mathbb{R}^+}(\omega_0+\tau_0)|\hat{\phi}_0|^2dx + \varepsilon\int_{\mathbb{R}^+} k_0\hat{\phi}_2\hat{\phi}_0 dx + \varepsilon^2\int_{\mathbb{R}^+} x\partial_x k_0\varphi_2\hat{\phi}_0 dx \\
&+ \varepsilon^2\int_{\mathbb{R}^+} x\partial_x J_0\hat{\phi}_0 dx + \int_{\mathbb{R}^+}(\omega_1+\tau_1)\hat{\phi}_2\hat{\phi}_0 dx \\
&- \int_{\mathbb{R}^+} x\partial_x(\omega_1+\tau_1)\varphi_2\hat{\phi}_0 dx + \sum_{i=1}^{3} I_i.
\end{aligned} \tag{3.2.25}$$

where

$$\begin{aligned}
I_1 &= b_0\varepsilon^2\int_{\mathbb{R}^+} x\partial_x(\partial_x^2\varphi_0)\hat{\phi}_0 dx, \\
I_2 &= 2\varepsilon^2\int_{\mathbb{R}^+} k_0 x\partial_x^2\varphi_2\cdot\hat{\phi}_0 dx, \\
I_3 &= -\int_{\mathbb{R}^+} xu\partial_x^2\varphi_0\cdot\hat{\phi}_0 dx.
\end{aligned}$$

which can be estimated as follows:

$$\begin{aligned}
I_1 &= b_0\varepsilon^2 \int_{\mathbb{R}^+} (\partial_x \hat{\phi}_0 - 2\partial_x^2 \varphi_0)\hat{\phi}_0 dx \\
&= -b_0\varepsilon^2 \int_{\mathbb{R}^+} |\partial_x \hat{\phi}_0|^2 dx + 2b_0\varepsilon^2 \int_{\mathbb{R}^+} \partial_x^2 \varphi_0 \cdot \hat{\phi}_0 dx \\
&= -b_0\varepsilon^2 \int_{\mathbb{R}^+} |\partial_x \hat{\phi}_0|^2 dx - 2b_0\varepsilon^2 \int_{\mathbb{R}^+} \partial_x \varphi_0 \partial_x \hat{\phi}_0 dx, \qquad (3.2.26)
\end{aligned}$$

$$\begin{aligned}
|I_2| \leq& C\varepsilon^2 \left| \int_{\mathbb{R}^+} x\partial_x \varphi_2 \partial_x \hat{\phi}_0 dx \right| + C\varepsilon^2 \left| \int_{\mathbb{R}^+} \partial_x \varphi_2 \hat{\phi}_0 dx \right| \\
\leq& C\varepsilon^2 \left| \int_{\mathbb{R}^+} \hat{\phi}_2 \partial_x \hat{\phi}_0 dx \right| + C\varepsilon^2 \left| \int_{\mathbb{R}^+} \varphi_2 \partial_x \hat{\phi}_0 dx \right|, \qquad (3.2.27)
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= -\int_{\mathbb{R}^+} x^2 u \partial_x^2 \varphi_0 \partial_x \varphi_0 dx \\
&= \int_{\mathbb{R}^+} xu|\partial_x \varphi_0|^2 dx + \int_{\mathbb{R}^+} x^2 \partial_x u |\partial_x \varphi_0|^2 dx \\
&= \int_{\mathbb{R}^+} u(x\partial_x \varphi_0)\partial_x \varphi_0 dx + \int_{\mathbb{R}^+} \partial_x u \cdot |x\partial_x \varphi_0|^2 dx \\
&= \int_{\mathbb{R}^+} u\partial_x \varphi_0 \hat{\phi}_0 dx + \int_{\mathbb{R}^+} \partial_x u |\hat{\phi}_0|^2 dx. \qquad (3.2.28)
\end{aligned}$$

So,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\hat{\phi}_0\|^2 + b_0\varepsilon^2 \int_{\mathbb{R}^+} |\partial_x \hat{\phi}_0|^2 dx \\
\leq& C\int_{\mathbb{R}^+} |\varphi_0|^2 dx + C\int_{\mathbb{R}^+} |\varphi_2|^2 dx + C\int_{\mathbb{R}^+} |\hat{\phi}_0|^2 dx + C\int_{\mathbb{R}^+} |\hat{\phi}_2|^2 dx \\
&+ C\int_{\mathbb{R}^+} |\partial_x \varphi_0|^2 dx + \frac{b_0\varepsilon^2}{2}\int_{\mathbb{R}^+} |\partial_x \hat{\phi}_0|^2 dx + C\varepsilon^4 \int_{\mathbb{R}^+} |x\partial_x J_0|^2 dx,
\end{aligned} \qquad (3.2.29)$$

i.e.

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\hat{\phi}_0\|^2 + C\varepsilon^2 \int_{\mathbb{R}^+} |\partial_x \hat{\phi}_0|^2 dx \\
\leq& C\int_{\mathbb{R}^+} |\varphi_0|^2 dx + C\int_{\mathbb{R}^+} |\varphi_2|^2 dx + C\int_{\mathbb{R}^+} |\hat{\phi}_0|^2 dx
\end{aligned}$$

$$+ C\int_{\mathbb{R}^+}|\hat{\phi}_2|^2dx + C\int_{\mathbb{R}^+}|\partial_x\varphi_0|^2dx + C\varepsilon^4\int_{\mathbb{R}^+}|x\partial_xJ_0|^2dx. \tag{3.2.30}$$

Next, it follows from (3.1.16) that $\hat{\phi}_1$ solves

$$\begin{aligned}
&\beta\partial_t\hat{\phi}_1 + \partial_x(\beta u)\hat{\phi}_1 + x\partial_x\beta\partial_t\varphi_1 + x\beta u\partial_x^2\varphi_1 + x\partial_x^2\varphi_2\\
&+ x\partial_x(\beta\tau_2)\varphi_2 + (\beta\tau_2)\hat{\phi}_0 + x\partial_x(\beta\tau_3)\varphi_2 + \beta\tau_3\hat{\phi}_2\\
=&\varepsilon^2x\partial_x(\beta b_2)\partial_x^2\varphi_1 + \varepsilon^2\beta b_2x\partial_x(\partial_x^2\varphi_1) + \varepsilon^2x\partial_x(\beta J_1).
\end{aligned} \tag{3.2.31}$$

We multiply the above equation by $\hat{\phi}_1$ and integrate the resulting equation over $\mathbb{R}^+$ to obtain after integration by parts that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^+}\beta|\hat{\phi}_1|^2dx - \frac{1}{2}\int_{\mathbb{R}^+}\partial_t\beta|\hat{\phi}_1|^2dx + \int_{\mathbb{R}^+}x\partial_x\beta\partial_t\varphi_1\cdot\hat{\phi}_1dx\\
&+ \int_{\mathbb{R}^+}\partial_x(\beta u)|\hat{\phi}_1|^2dx + \int_{\mathbb{R}^+}x\partial_x^2\varphi_2\cdot\hat{\phi}_1dx + \int_{\mathbb{R}^+}x\partial_x(\beta\tau_2)\varphi_2\hat{\phi}_1dx\\
&+ \int_{\mathbb{R}^+}(\beta\tau_2)\hat{\phi}_0\hat{\phi}_1dx + \int_{\mathbb{R}^+}x\partial_x(\beta\tau_3)\varphi_2\hat{\phi}_1dx + \int_{\mathbb{R}^+}(\beta\tau_3)\hat{\phi}_2\hat{\phi}_1dx\\
\le&\sum_{j=4}^{6}I_i + \varepsilon^2\int_{\mathbb{R}^+}x\partial_x\beta J_1\hat{\phi}_1dx + \varepsilon^2\int_{\mathbb{R}^+}\beta x\partial_xJ_1\cdot\hat{\phi}_1dx,
\end{aligned} \tag{3.2.32}$$

where

$$\begin{aligned}
I_4 &= -\int_{\mathbb{R}^+}x\beta u\partial_x^2\varphi_1\hat{\phi}_1dx,\\
I_5 &= \varepsilon^2\int_{\mathbb{R}^+}\beta b_2x\partial_x(\partial_x^2\varphi_1)\hat{\phi}_1dx,\\
I_6 &= \varepsilon^2\int_{\mathbb{R}^+}x\partial_x(\beta b_2)\partial_x^2\varphi_1\cdot\hat{\phi}_1dx,
\end{aligned}$$

which can be estimated as follows:

$$\begin{aligned}
|I_4| &\le \left|\int_{\mathbb{R}^+}x^2u\partial_x^2\varphi_1\partial_x\varphi_1dx\right|\\
&\le \left|\int_{\mathbb{R}^+}xu(\partial_x\varphi_1)^2dx\right| + \left|\int_{\mathbb{R}^+}x^2\partial_xu(\partial_x\varphi_1)^2dx\right|\\
&\le \left|\int_{\mathbb{R}^+}u\partial_x\varphi_1\cdot\hat{\phi}_1dx\right| + \left|\int_{\mathbb{R}^+}\partial_xu(\hat{\phi}_1)^2dx\right|,
\end{aligned} \tag{3.2.33}$$

$$|I_5| \le C_2\varepsilon^2\left|\int_{\mathbb{R}^+}(\partial_x^2\hat{\phi}_1 - 2\partial_x^2\varphi_1)\cdot\hat{\phi}_1dx\right|$$

$$
\begin{aligned}
&\le C_2\varepsilon^2\left|\int_{\mathbb{R}^+}\partial_x^2\hat{\phi}_1\cdot\hat{\phi}_1dx-2\int_{\mathbb{R}^+}\partial_x^2\varphi_1)\cdot\hat{\phi}_1dx\right|\\
&\le C_2\varepsilon^2\int_{\mathbb{R}^+}\left|\partial_x\hat{\phi}_1\right|^2dx+2C_2\varepsilon^2\int_{\mathbb{R}^+}\left|\partial_x\varphi_1\partial_x\hat{\phi}_1\right|dx\\
&\le 2C_2\varepsilon^2\int_{\mathbb{R}^+}|\partial_x\hat{\phi}_1|^2dx+C_2\varepsilon^2\int_{\mathbb{R}^+}|\partial_x\varphi_1|^2\,dx, \qquad (3.2.34)
\end{aligned}
$$

where C_2 is a uniform constant which is independent of ε.

$$
\begin{aligned}
|I_6|&\le C\varepsilon^2\left|\int_{\mathbb{R}^+}\partial_x^2\varphi_1\cdot\hat{\phi}_1dx\right|\le C\varepsilon^2\int_{\mathbb{R}^+}\left|\partial_x\varphi_1\partial_x\hat{\phi}_1\right|dx\\
&\le C\varepsilon^2\int_{\mathbb{R}^+}|\partial_x\varphi_1|^2dx+C\varepsilon^2\int_{\mathbb{R}^+}|\partial_x\hat{\phi}_1|^2dx \qquad (3.2.35)
\end{aligned}
$$

So,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^+}\beta|\hat{\phi}_1|^2dx+\int_{\mathbb{R}^+}x\partial_x^2\varphi_2\hat{\phi}_1dx+C_1\varepsilon^2\int_{\mathbb{R}^+}|\partial_x^2\hat{\phi}_1|dx\\
\le&\frac{1}{2}\int_{\mathbb{R}^+}\partial_t\beta|\hat{\phi}_1|^2dx-\int_{\mathbb{R}^+}x\partial_x\beta\partial_t\varphi_1\cdot\hat{\phi}_1dx-\int_{\mathbb{R}^+}\partial_x(\beta u)|\hat{\phi}_1|^2dx\\
&-\int_{\mathbb{R}^+}x\partial_x(\beta\tau_2)\varphi_2\hat{\phi}_1dx-\int_{\mathbb{R}^+}(\beta\tau_2)\hat{\phi}_0\hat{\phi}_1dx-\int_{\mathbb{R}^+}x\partial_x(\beta\tau_3)\varphi_2\hat{\phi}_1dx\\
&-\int_{\mathbb{R}^+}(\beta\tau_3)\hat{\phi}_2\hat{\phi}_1dx+\left|\int_{\mathbb{R}^+}u\partial_x\varphi_1\cdot\hat{\phi}_1dx\right|+\left|\int_{\mathbb{R}^+}\partial_xu(\hat{\phi}_1)^2dx\right|\\
&C\varepsilon^2\int_{\mathbb{R}^+}|\partial_x\varphi_1|^2dx+C\varepsilon^2\int_{\mathbb{R}^+}|\partial_x\hat{\phi}_1|^2dx\\
\le&C\int_{\mathbb{R}^+}|\hat{\phi}_0|^2dx+C\int_{\mathbb{R}^+}|\hat{\phi}_1|^2dx+C\int_{\mathbb{R}^+}|\hat{\phi}_2|^2dx+C\int_{\mathbb{R}^+}|\varphi_2|^2dx\\
&+C\int_{\mathbb{R}^+}|\partial_t\varphi_1|^2dx+C\int_{\mathbb{R}^+}|\partial_x\varphi_1|^2dx+\frac{C_2}{2}\varepsilon^2\int_{\mathbb{R}^+}|\partial_x\hat{\phi}_1|^2dx\\
&+C\varepsilon^4\int_{\mathbb{R}^+}|J_1|^2dx+C\varepsilon^4\int_{\mathbb{R}^+}|x\partial_xJ_1|^2dx, \qquad (3.2.36)
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^+}\beta|\hat{\phi}_1|^2dx+C\varepsilon^2\int_{\mathbb{R}^+}|\partial_x\hat{\phi}_1|^2dx+\int_{\mathbb{R}^+}x\partial_x^2\varphi_2\hat{\phi}_1dx\\
\le&C\int_{\mathbb{R}^+}|\hat{\phi}_0|^2dx+C\int_{\mathbb{R}^+}|\hat{\phi}_1|^2dx+C\int_{\mathbb{R}^+}|\hat{\phi}_2|^2dx+C\int_{\mathbb{R}^+}|\varphi_2|^2dx\\
&+C\int_{\mathbb{R}^+}|\partial_t\varphi_1|^2dx+C\int_{\mathbb{R}^+}|\partial_x\varphi_1|^2dx+C\varepsilon^4\int_{\mathbb{R}^+}|J_1|^2dx+C\varepsilon^4\int_{\mathbb{R}^+}|x\partial_xJ_1|^2dx.
\end{aligned}
\qquad (3.2.37)
$$

Similarly, one can show that $\hat{\phi}_2$ satisfies

$$\begin{aligned}
&\delta\partial_t\hat{\phi}_2 + \partial_x(\delta u)\hat{\phi}_2 + x\partial_x\delta\cdot\partial_t\varphi_2 + xu\partial_x^2\varphi_1 + xu\delta\partial_x^2\varphi_2 + (\delta\omega_3+\delta\tau_5)\hat{\phi}_2\\
&+ x\partial_x(\delta\omega_2+\delta\tau_4)\varphi_0 + (\delta\omega_2+\delta\tau_4)\hat{\phi}_0 + x\partial_x(\delta\omega_3+\delta\tau_5)\varphi_2\\
=&\varepsilon^2\delta k_3\hat{\phi}_2 + \varepsilon^2 x\partial_x(\delta k_3)\varphi_2 + \varepsilon^2 x\partial_x\delta\cdot J_2 + \varepsilon^2\delta\cdot x\partial_x J_2. \qquad (3.2.38)
\end{aligned}$$

Taking inner product in $L^2(\mathbb{R}^+)$ with $\hat{\phi}_2$, and integrating give us the following estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^+}\delta|\hat{\phi}_2|^2dx + \int_{\mathbb{R}^+}x\partial_x\delta\partial_t\varphi_2\cdot\hat{\phi}_2dx + \int_{\mathbb{R}^+}\partial_x(\delta u)|\hat{\phi}_2|^2dx\\
&+\int_{\mathbb{R}^+}x\partial_x^2\varphi_1\cdot\hat{\phi}_2dx + \int_{\mathbb{R}^+}x\partial_x(\delta\omega_2+\delta\tau_4)\varphi_0\hat{\phi}_2dx\\
&+\int_{\mathbb{R}^+}(\delta\omega_2+\delta\tau_4)\hat{\phi}_0\hat{\phi}_2dx + \int_{\mathbb{R}^+}x\partial_x(\delta\omega_3+\delta\tau_5)\varphi_2\hat{\phi}_2dx + \int_{\mathbb{R}^+}(\delta\omega_3+\delta\tau_5)|\hat{\phi}_2|^2dx\\
=&\int_{\mathbb{R}^+}x\delta\partial_x^2\varphi_2\hat{\phi}_2dx + \varepsilon^2\int_{\mathbb{R}^+}\delta k_3|\hat{\phi}_2|^2dx + \frac{1}{2}\int_{\mathbb{R}^+}\partial_t\delta|\hat{\phi}_2|^2dx\\
&+\varepsilon^2\int_{\mathbb{R}^+}x\partial_x(\delta k_3)\varphi_2\hat{\phi}_2dx + \varepsilon^2\int_{\mathbb{R}^+}x\partial_x\delta\cdot J_2\hat{\phi}_2dx + \varepsilon^2\int_{\mathbb{R}^+}\delta x\partial_x J_2\hat{\phi}_2dx.
\end{aligned}\qquad (3.2.39)$$

Note that

$$\begin{aligned}
&\int_{\mathbb{R}^+}x\delta\partial_x^2\varphi_2\hat{\phi}_2dx\\
&=-\int_{\mathbb{R}^+}\partial_x(\delta u)|\hat{\phi}_2|^2dx - \int_{\mathbb{R}^+}\delta u\partial_x\varphi_2\hat{\phi}_2dx\\
&=-\int_{\mathbb{R}^+}\partial_x(\delta u)|\hat{\phi}_2|^2dx - \int_{\mathbb{R}^+}(\delta u(x,t)-\delta u(0,t))\partial_x\varphi_2\hat{\phi}_2dx\\
&=-\int_{\mathbb{R}^+}\partial_x(\delta u)|\hat{\phi}_2|^2dx - \int_{\mathbb{R}^+}\partial_x(\delta u)(\xi,t)|\hat{\phi}_2|^2dx,\ \xi\in(0,x). \qquad (3.2.40)
\end{aligned}$$

So,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^+}\delta|\hat{\phi}_2|^2dx + \int_{\mathbb{R}^+}x\partial_x^2\varphi_1\hat{\phi}_2dx\\
\le& C\int_{\mathbb{R}^+}|\varphi_0|^2dx + C\int_{\mathbb{R}^+}|\varphi_2|^2dx + C\int_{\mathbb{R}^+}|\hat{\phi}_0|^2dx + C\int_{\mathbb{R}^+}|\hat{\phi}_2|^2dx\\
&+C\int_{\mathbb{R}^+}|\partial_t\varphi_2|^2dx + C\varepsilon^4\int_{\mathbb{R}^+}|J_2|^2dx + C\varepsilon^4\int_{\mathbb{R}^+}|x\partial_x J_2|^2dx. \qquad (3.2.41)
\end{aligned}$$

Summing (3.2.30), (3.2.37) and (3.2.41) up and noticing that

$$\begin{aligned}
\int_{\mathbb{R}^+} x\partial_x^2\varphi_1\hat{\phi}_2 dx &= -\int_{\mathbb{R}^+} \partial_x\varphi_1\hat{\phi}_2 dx - \int_{\mathbb{R}^+} x\partial_x\varphi_1\partial_x\hat{\phi}_2 dx \\
&= -\int_{\mathbb{R}^+} \partial_x\varphi_1\cdot\hat{\phi}_2 dx - \int_{\mathbb{R}^+} \hat{\phi}_1\partial_x\hat{\phi}_2 dx \\
&= -\int_{\mathbb{R}^+} \partial_x\varphi_1\cdot\hat{\phi}_2 dx - \int_{\mathbb{R}^+} \hat{\phi}_1\partial_x(x\partial_x\varphi_2) dx \\
&= -\int_{\mathbb{R}^+} \partial_x\varphi_1\cdot\hat{\phi}_2 dx - \int_{\mathbb{R}^+} x\hat{\phi}_1\partial_x^2\varphi_2 dx - \int_{\mathbb{R}^+} \hat{\phi}_1\partial_x\varphi_2 dx \\
&= -\int_{\mathbb{R}^+} \partial_x\varphi_1\cdot\hat{\phi}_2 dx - \int_{\mathbb{R}^+} x\partial_x^2\varphi_2\cdot\hat{\phi}_1 dx - \int_{\mathbb{R}^+} x\partial_x\varphi_1\partial_x\varphi_2 dx \\
&= -2\int_{\mathbb{R}^+} \partial_x\varphi_1\cdot\hat{\phi}_2 dx - \int_{\mathbb{R}^+} x\partial_x^2\varphi_2\cdot\hat{\phi}_1 dx,
\end{aligned}$$

together with Lemma 3.2.1, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\hat{\phi}(t)\|^2_{\mathcal{A}_0(t)} + C\varepsilon^2\sum_{j=0}^{1}\|\partial_x\hat{\phi}_j(t)\|^2 \\
\leq &C\|\hat{\phi}(t)\|^2 + C\|\varphi\|^2 + C\|\partial_t\varphi(t)\|^2 \\
&+ C\sum_{j=0}^{1}\|\partial_x\varphi_j(t)\|^2 + C\varepsilon^4\|J(\varepsilon,t)\|^2 + C\varepsilon^4\|x\partial_x J(\varepsilon,t)\|^2 \\
\leq &C\|\hat{\phi}(t)\|^2 + C\|\partial_t\varphi(t)\|^2 + C\varepsilon^3 + C\varepsilon^5. \qquad (3.2.42)
\end{aligned}$$

Due to (3.2.6), (3.1.7) and (3.1.10) the above inequality is equivalent to

$$\frac{1}{2}\frac{d}{dt}\|\hat{\phi}(t)\|^2 + C\varepsilon^2\sum_{j=0}^{1}\|\partial_x\hat{\phi}_j(t)\|^2 \leq C\|\hat{\phi}(t)\|^2 + C\|\partial_t\varphi(t)\|^2 + C\varepsilon^3. \qquad (3.2.43)$$

which implies

$$\frac{1}{2}\frac{d}{dt}\|x\partial_x\varphi(t)\|^2 + C\varepsilon^2\sum_{j=0}^{1}\|\partial_x(x\partial_x\varphi_j)(t)\|^2 \leq C\|x\partial_x\varphi(t)\|^2 + C\|\partial_t\varphi(t)\|^2 + C\varepsilon^3. \qquad (3.2.44)$$

□

Combining Lemma 3.2.2 and Lemma 3.2.3 yield the following Proposition for the estimate of the tangential derivatives $D_{tan}\varphi$.

Proposition 3.2.4 *It holds that*

$$\sup_{0\le t\le T} \|D_{tan}\varphi(t)\|^2 + C\varepsilon^2 \sum_{j=0}^{1} \int_0^T \|\partial_x D_{tan}\varphi(t)\|^2 dt \le C\varepsilon^3, \tag{3.2.45}$$

where C is the same as in Lemma 3.2.3.

Proof: Add (3.2.8) and (3.2.44), we have

$$\frac{d}{dt}\|D_{tan}\varphi(t)\|^2 + C\varepsilon^2 \sum_{j=0}^{1} \|\partial_x D_{tan}\varphi(t)\|^2 \le C\|D_{tan}\varphi(t)\|^2 + C\varepsilon^3. \tag{3.2.46}$$

In view of the fact that both ψ and $\hat{\phi}$ satisfy the zero initial conditions, using Gronwall's inequality yields

$$\|D_{tan}\varphi(t)\|^2 \le C\varepsilon^3. \tag{3.2.47}$$

Then combining (3.2.46), one shows that

$$\|D_{tan}\varphi(t)\|^2 + C\varepsilon^2 \sum_{j=0}^{1} \int_0^T \|\partial_x D_{tan}\varphi(t)\|^2 dt \le C\varepsilon^3. \tag{3.2.48}$$

Taking supreme with respect to time t implies the desired conclusion. □

3.2.3 Normal Derivatives Estimates

It remains to estimate the normal derivatives of φ. We have the following lemma.

Lemma 3.2.5 *The normal derivatives of φ satisfy*

$$\sup_{0\le t\le T} \sum_{j=0}^{1} \|\partial_x \varphi_j\|^2 \le C\varepsilon^2, \tag{3.2.49}$$

and

$$\sup_{0\le t\le T} \|\partial_x \varphi_2\|^2 \le C\varepsilon. \tag{3.2.50}$$

where C is a uniform constant depending on the bounds of $V', \partial_x V', \partial_t V', \partial^2_{xt} V', c_0, c_1$ and $x\partial_x V'$.

Proof: From (3.2.5) and using Young's inequality, we obtain

$$\begin{aligned}
\varepsilon^2 \sum_{j=0}^{1} \|\partial_x \varphi_j\|^2 &\le -\frac{1}{2}\frac{d}{dt}\|\varphi(t)\|^2_{\mathcal{A}_0(t)} + C\|\varphi(t)\|^2 + \varepsilon^4 \|J(\varepsilon,t)\|^2 \\
&\le C\big(\varepsilon\|\partial_t \varphi(t)\| + \varepsilon^{-1}\|\varphi(t)\| + \|\varphi(t)\|\big) + C\varepsilon^5 \\
&\le C\varepsilon^4 + C\varepsilon^5, \qquad (3.2.51)
\end{aligned}$$

where we have used Lemma 3.2.1 and Proposition 3.2.4. Thus

$$\sup_{0\le t\le T} \sum_{j=0}^{1} \|\partial_x \varphi_j\|^2 \le C\varepsilon^2. \qquad (3.2.52)$$

We now turn to estimate $\partial_x \varphi_2$, which will be denoted by $\breve{\varphi}_2$. Differentiating the equation (3.1.17) with respect to x and combing the equation (3.1.16), we derive that

$$\begin{aligned}
&\delta\partial_t \breve{\varphi}_2 + \delta u \partial_x \breve{\varphi}_2 + (\delta\omega_3 + \delta\tau_5)\breve{\varphi}_2 + \frac{1}{\varepsilon^2 \beta b_2}\breve{\varphi}_2 + \partial_x(\delta u)\breve{\varphi}_2 - \varepsilon^2 \delta k_3 \breve{\varphi}_2 \\
&= \mathcal{Q}(\varepsilon, x, t), \qquad (3.2.53)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{Q}(\varepsilon,x,t) = &-\frac{1}{\varepsilon^2 \beta b_2}\big(\beta\partial_t\varphi_1 + \beta u\partial_x\varphi_1 + \beta\tau_2\varphi_0 + \beta\tau_3\varphi_2 - \varepsilon^2\beta J_1\big) \\
&- \partial_x\delta\partial_t\varphi_2 - \partial_x(\delta\omega_2 + \delta\tau_4)\varphi_0 - (\delta\omega_2 + \delta\tau_4)\partial_x\varphi_0 \\
&- \partial_x(\delta\omega_3 + \delta\tau_5)\varphi_2 + \varepsilon^2\partial_x(\delta k_3)\varphi_2 + \varepsilon^2\partial_x\delta\cdot J_2 + \varepsilon^2\delta\partial_x J_2.
\end{aligned}$$

Multiplying $\breve{\varphi}_2$ and integrating by parts give

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^+} \delta|\breve{\varphi}_2|^2 dx - \frac{1}{2}\int_{\mathbb{R}^+} \partial_t\delta|\breve{\varphi}_2|^2 dx + \frac{1}{2}\int_{\mathbb{R}^+} \partial_x(\delta u)|\breve{\varphi}_2|^2 dx \\
&+ \int_{\mathbb{R}^+} (\delta\omega_3 + \delta\tau_5)|\breve{\varphi}_2|^2 dx + \frac{1}{\varepsilon^2\beta b_2}\int_{\mathbb{R}^+} |\breve{\varphi}_2|^2 dx - \varepsilon^2\int_{\mathbb{R}^+} \delta k_3|\breve{\varphi}_2|^2 dx \\
= &\int_{\mathbb{R}^+} \mathcal{Q}(\varepsilon,x,t)\breve{\varphi}_2 dx.
\end{aligned}$$

where

$$\int_{\mathbb{R}^+} \mathcal{Q}(\varepsilon,x,t)\breve{\varphi}_2 dx = \int_{\mathbb{R}^+} \big(\varepsilon\mathcal{Q}(\varepsilon,x,t)\big)\big(\frac{1}{\varepsilon}\breve{\varphi}_2\big) dx$$

$$
\begin{aligned}
=&-\int_{\mathbb{R}+}\frac{1}{\varepsilon b_2}\partial_t\varphi_1(\frac{1}{\varepsilon}\breve{\varphi}_2)dx-\int_{\mathbb{R}+}\frac{u'}{\varepsilon b_2}\partial_x\varphi_1(\frac{1}{\varepsilon}\breve{\varphi}_2)dx-\int_{\mathbb{R}+}\frac{\tau_2}{\varepsilon b_2}\varphi_0(\frac{1}{\varepsilon}\breve{\varphi}_2)dx\\
&-\int_{\mathbb{R}+}\frac{\tau_3}{\varepsilon b_2}\varphi_2\breve{\varphi}_2dx+\int_{\mathbb{R}+}\varepsilon J_1\varphi_2(\frac{1}{\varepsilon}\breve{\varphi}_2)dx-\int_{\mathbb{R}+}\partial_x(\delta\omega_3+\delta\tau_5)\varphi_2\breve{\varphi}_2dx\\
&-\int_{\mathbb{R}+}\partial_x\delta\partial_t\varphi_2\breve{\varphi}_2dx-\int_{\mathbb{R}+}\partial_x(\delta\omega_2+\delta\tau_4)\varphi_0\breve{\varphi}_2dx-\int_{\mathbb{R}+}\partial_x(\delta\omega_2+\delta\tau_4)\partial_x\varphi_0\cdot\breve{\varphi}_2dx\\
&+\varepsilon^2\int_{\mathbb{R}+}\partial_x(\delta k_3)\varphi_2\breve{\varphi}_2dx+\varepsilon^2\int_{\mathbb{R}+}\partial_x\delta\cdot J_2\breve{\varphi}_2dx+\varepsilon^2\int_{\mathbb{R}+}\delta\partial_xJ_2\breve{\varphi}_2dx\\
\le&\frac{C}{\varepsilon^2}\int_{\mathbb{R}+}|\partial_t\varphi_1|^2dx+\frac{1}{2\beta b_2\varepsilon^2}\int_{\mathbb{R}+}|\breve{\varphi}_2|^2dx+\frac{C}{\varepsilon^2}\int_{\mathbb{R}+}|\partial_x\varphi_1|^2dx+C\int_{\mathbb{R}+}|\partial_t\varphi_2|^2dx\\
&+\frac{C}{\varepsilon^2}\int_{\mathbb{R}+}|\varphi_0|^2dx+\varepsilon^2\int_{\mathbb{R}+}|J_1|^2dx+C\int_{\mathbb{R}+}|\partial_t\varphi_1|^2dx+C\int_{\mathbb{R}+}|\varphi_0|^2dx\\
&+C\varepsilon^4\int_{\mathbb{R}+}|\varphi_2|^2dx+C\int_{\mathbb{R}+}|\varphi_2|^2dx+C\int_{\mathbb{R}+}|\partial_x\varphi_0|^2dx\\
&+C\varepsilon^4\int_{\mathbb{R}+}|J_2|^2dx+C\varepsilon^4\int_{\mathbb{R}+}|\partial_xJ_2|^2dx. \qquad (3.2.54)
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}+}\delta|\breve{\varphi}_2|^2dx+\frac{C}{\varepsilon^2}\int_{\mathbb{R}+}|\breve{\varphi}_2|^2dx\\
\le&C\int_{\mathbb{R}+}|\breve{\varphi}_2|^2dx+\frac{C}{\varepsilon^2}\int_{\mathbb{R}+}|\varphi_0|^2dx+\frac{C}{\varepsilon^2}\int_{\mathbb{R}+}|\partial_t\varphi_1|^2dx+\frac{C}{\varepsilon^2}\int_{\mathbb{R}+}|\partial_x\varphi_1|^2dx\\
&+\varepsilon^2\int_{\mathbb{R}+}|J_1|^2dx+C\int_{\mathbb{R}+}|\partial_t\varphi_2|^2dx+C\int_{\mathbb{R}+}|\partial_x\varphi_0|^2dx\\
&+C\int_{\mathbb{R}+}|\varphi_2|^2dx+C\varepsilon^4\int_{\mathbb{R}+}|J_2|^2dx+C\varepsilon^4\int_{\mathbb{R}+}|\partial_xJ_2|^2dx\le C\int_{\mathbb{R}+}|\breve{\varphi}_2|^2dx+C\varepsilon,
\end{aligned}
\qquad (3.2.55)
$$

where we have used (1.2.1), Lemma 3.2.1, Proposition 3.2.4, and the properties of the remainder term $J(\varepsilon,x,t)$ (3.1.7) and (3.1.8). Since $\breve{\varphi}_2(x,0)=0$, then by Gronwall's inequality, we get

$$
\|\breve{\varphi}_2\|^2\le C\varepsilon. \qquad (3.2.56)
$$

Thus the proof of Lemma 3.2.5 is completed. □

3.3 Pointwise Estimates

Now, we can show the pointwise estimate for the error terms. To this end, we need the following Sobolev's inequality.

Lemma 3.3.1 *Let $g(x)$ be a function defined in $\mathbb{R}^+$ such that $g, \partial_x g$ are in $L^2(\mathbb{R}^+)$. Then the following inequality is valid for g,*

$$\sup_{0\le t\le T} |g(x,t)| \le \sqrt{2} \sup_{0\le t\le T} \|g(x,t)\|^{1/2} \cdot \|\partial_x g(x,t)\|^{1/2}. \tag{3.3.1}$$

Thus we have the estimate for any ε such that $0 < \varepsilon < 1$:

Proposition 3.3.2

$$\sup_{\mathbb{R}^+\times[0,T]} |\varphi_0(x,t)| \le C\varepsilon^{7/4}, \tag{3.3.2}$$

$$\sup_{\mathbb{R}^+\times[0,T]} |\varphi_1(x,t)| \le C\varepsilon^{7/4}, \tag{3.3.3}$$

and

$$\sup_{\mathbb{R}^+\times[0,T]} |\varphi_2(x,t)| \le C\varepsilon^{3/2}. \tag{3.3.4}$$

Proof: As a consequence of Lemma 3.2.1, Proposition 3.2.4, and Lemma 3.3.1, we obtain that

$$\begin{aligned} &\sup_{\mathbb{R}^+\times[0,T]} |\varphi_2(x,t)| \\ \le&\sqrt{2} \sup_{0\le t\le T} \|\varphi_2^0(x,t)\|^{1/2} \|\partial_x \varphi_2(x,t)\|^{1/2} \le C\varepsilon^{3/2}, \end{aligned} \tag{3.3.5}$$

and

$$\begin{aligned} &\sup_{\mathbb{R}^+\times[0,T]} |\varphi_j(x,t)| \\ \le&\sqrt{2} \sup_{0\le t\le T} \|\varphi_j(x,t)\|^{1/2} \|\partial_x \varphi_j(x,t)\|^{1/2} \le C\varepsilon^{7/4}, \quad j = 0, 1. \end{aligned} \tag{3.3.6}$$

□

We next return to the original notations of the problem (??). Define the correction term $\mathcal{R}^\varepsilon$ as

$$\mathcal{R}^\varepsilon = (\mathcal{R}_0^\varepsilon, \mathcal{R}_1^\varepsilon, \mathcal{R}_2^\varepsilon)^T$$

$$= \sum_{i=1}^{2} \varepsilon^i a^i(x,t) + V_{bd}(\varepsilon, \frac{x}{\varepsilon}, t). \tag{3.3.7}$$

and

$$E^\varepsilon = (\frac{1}{R+C_v}\mathcal{R}_0^\varepsilon, \frac{1}{2}\mathcal{R}_1^\varepsilon, \frac{1}{4R}\mathcal{R}_2^\varepsilon)^T. \tag{3.3.8}$$

Then it follows from the previous discussions, the following uniform stability result for the linearized Navier-Stokes equations (1.2.2)–(1.2.4) holds, which is the main result of this paper:

Theorem 3.3.3 *Let $m \geq 13$ be an integer. Suppose that the initial data $V_0 \in H^m(\mathbb{R}^+)$ satisfies the compatibility condition of order $\left[\frac{m}{2}\right] - 1$ for (1.2.2)–(1.2.4) for any $\varepsilon > 0$ and the compatibility condition of order $[m] - 1$ for (1.2.10)–(1.2.12). Then the solution V^ε of the initial boundary value problem of the linearized Navier-Stokes equations (1.2.2)–(1.2.4), the solution V^0 of the initial boundary value problem of the linearized Euler equations (1.2.10)–(1.2.12), and the correcting term E^ε defined in (3.3.8) exist uniquely in the spaces such that*

$$V^\varepsilon \in \bigcap_{j=0}^{\left[\frac{m}{2}\right]} C^j([0,T]; H^{m-2j}(\mathbb{R}^+)), \tag{3.3.9}$$

$$V^0 \in \bigcap_{j=0}^{[m]} C^j([0,T]; H^{m-j}(\mathbb{R}^+)), \tag{3.3.10}$$

$$E^\varepsilon \in \bigcap_{j=0}^{\left[\frac{m}{2}\right]-15} C^j([0,T]; H^{m-15-j}(\mathbb{R}^+)), \tag{3.3.11}$$

and the following estimates hole for any ε such that $0 < \varepsilon < 1$:

$$\sup_{(x,t)\in\mathbb{R}^+\times[0,T]} \left|\varrho^\varepsilon - \varrho^0 - E_2^\varepsilon\right| \leq C\varepsilon^{\frac{3}{2}}, \tag{3.3.12}$$

$$\sup_{(x,t)\in\mathbb{R}^+\times[0,T]} \left|u^\varepsilon - u^0 - E_1^\varepsilon\right| \leq C\varepsilon^{\frac{7}{4}}, \tag{3.3.13}$$

and

$$\sup_{(x,t)\in\mathbb{R}^+\times[0,T]} \left|\theta^\varepsilon - \theta^0 - E_0^\varepsilon\right| \leq C\varepsilon^{\frac{7}{4}}, \tag{3.3.14}$$

where C is a uniform constant depending on $\sup_{\mathbb{R}^+\times[0,T]}(|V'|, |\partial_t V'|, |\partial_x V'|, |\partial_{xt}^2 V'|, |x\partial_x V'|)$.

As a consequence from the Theorem 3.3.3, we have

Corollary 3.3.4 *Under the assumptions in Theorem* 3.3.3, *there holds the estimate for any* $\eta > 0$, $0 < \sigma < 1$, *and* $0 < \varepsilon < 1$,

$$\sup_{0\le t\le T, h\varepsilon^{1-\sigma}\le x}\Big\{\big|\varrho^\varepsilon(x,t)-\varrho^0(x,t)\big|+\big|u^\varepsilon(x,t)-u^0(x,t)\big| + \big|\theta^\varepsilon(x,t)-\theta^0(x,t)\big|\Big\}\le C_h\varepsilon, \tag{3.3.15}$$

where C_η *is a constant depending only on* η *and* $\sup_{\mathbb{R}+\times[0,T]}(|V'|, |\partial_t V'|, |\partial_x V'|, |\partial^2_{xt} V'|, |x\partial_x V'|)$.

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